Volume 70B
Number 1
Jan.–Mar. 1966

Mathematics and Mathematical Physics

U.S. DEPARTMENT OF COMMERCE

SECTION B

JOURNAL OF RESEARCH

NATIONAL BUREAU OF STANDARDS

Journal of Research

of the

National Bureau of Standards

B. Mathematics and Mathematical Physics

JAN.–MAR. • 1966

VOLUME • 70B

NUMBER • 1

Editor: Chester H. Page

Associate Editor: John J. Sopka

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1966

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Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 19, 1961).

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

Invariant Properties of the Spheroidal Potential of an Oblate Planet*

John P. Vinti[1]

(November 17, 1965)

The author's gravitational potential for an oblate planet, expressible in terms of oblate spheroidal coordinates, can be generalized by means of a metric-preserving transformation of the associated Cartesian system. This preserves separability of the problem of orbital motion when the potential coefficients $J_{2,1}$ and J_3 are taken into account. The inclusion of $J_{2,1}$ is not of practical importance, but has a clear physical interpretation. The inclusion of J_3, however, is of considerable practical importance, permitting a more accurate treatment than that given by perturbation theory. On the other hand, its physical significance is less clear, since including J_3 depends on translating the origin of spheroidal coordinates by a distance $\delta=\frac{1}{2}J_2^{-1}|J_3|$ equatorial radii. This distance, amounting to 7 km in the case of the earth, is much greater than any change in the geoid produced by J_3. It is clearly related to the long-periodic terms of perturbation theory and turns out to be equal to the displacement by J_3 of the plane of symmetry of those exactly elliptical polar orbits which are possible solutions with the spheroidal model.

Key Words: Invariant properties, spheroidal potential, oblate planet, metric-preserving transformation.

1. Introduction

If one neglects the axial asymmetry of an oblate planet, the gravitational potential V outside it is given by

$$V=-\frac{\mu}{r}\left[1-\sum_{n=1}^{\infty}\left(\frac{r_e}{r}\right)^n J_n P_n(\sin\theta)\right]. \tag{1}$$

Here μ is the product of the gravitational constant and the planet's mass, r_e is its equatorial radius, and r and θ are the planetocentric distance and latitude of a field point. The P_n's are the Legendre polynomials and the J_n's are the coefficients of potential, with $J_2\approx 10^{-3}$ and all higher J's $\approx 10^{-6}$ or smaller for the earth. The first harmonic, $n=1$, can be eliminated from (1) by taking the origin at the center of mass.

If $\mathbf{r}$ is the position vector of a satellite of such a planet, its motion around the planet is described by $\ddot{\mathbf{r}}=-\nabla V$. In earlier papers[2] [V1959a, b] I discovered a choice of J_n's that reduces the solution of this differential equation to a separable problem. This choice is

$$J_{2k}=(-1)^{k+1}J_2^k \qquad J_{2k+1}=(-1)^k J_1 J_2^k, \tag{2}$$

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[2] For convenience I shall use the Harvard system for references. Since there will be so many references to my own papers, however, I shall abbreviate my name to the initial V in the body of the paper.

corresponding to the spheroidal potential

$$V=-\frac{\mu(\rho+\eta\delta)}{\rho^2+c^2\eta^2}, \tag{3}$$

ρ, η, and φ being spheroidal coordinates. The relations among rectangular, spherical, and spheroidal coordinates are

$$x+iy=r\cos\theta\exp i\varphi=[(\rho^2+c^2)(1-\eta^2)]^{1/2}\exp i\varphi \tag{4.1}$$

$$z=r\sin\theta=\rho\eta. \tag{4.2}$$

Here

$$c^2=r_e^2J_2 \qquad \delta=-r_eJ_1 \tag{5}$$

and δ is the distance, taken positive northward, from the origin to the center of mass. Since $\delta=0$ when they coincide, this potential is symmetric with respect to the equatorial plane. Inversion of the separated equations led to an accurate reference orbit that takes into account the zeroth and second harmonic exactly and about two-thirds of the fourth harmonic [V1961a, b, 1962].

Shortly after the publication of V1959a, Brouwer and Pines [Brouwer and Clemence 1961, pp. 573, 592] found that the potential (3), with $\delta=0$, could be discovered in another way. If one imagines half the mass of the planet placed on the polar axis at $z=c'$ and half the mass at $z=-c'$, the corresponding two-center potential is

$$V=-\frac{1}{2}\frac{\mu}{r_1}-\frac{1}{2}\frac{\mu}{r_2}, \tag{6}$$

where

$$\begin{pmatrix} r_1^2 \\ r_2^2 \end{pmatrix}=x^2+y^2+(z\mp c')^2. \tag{6.1}$$

This corresponds to the separable problem of two fixed centers. If one now replaces c' by $c\sqrt{-1}$, the expression (6) remains real and leads to the separating potential (3) of the oblate planet. The significance of this procedure is traceable to the imaginary transformation that connects prolate and oblate spheroidal coordinates.

Some time later Aksenov, Grebenikov, and Demin [1963] discovered that if μ_1, μ_2, r_1, and r_2 are all complex, with $\mu_1r_1^{-1}$ and $\mu_2r_2^{-1}$ conjugate, the potential

$$V=-\frac{\mu_1}{r_1}-\frac{\mu_2}{r_2} \tag{7}$$

also leads to separability and enables one to fit not only μ and J_2 in (1) but also J_3, with the origin still located at the center of mass. An endeavor to understand this possibility in more physical terms led to the present paper.

2. A Further Summary of Previous Results

Let

$$\xi\equiv\rho/c, \tag{8}$$

where c is the adjustable distance in (4.1). The potential (3) was the most general function of oblate-spheroidal coordinates that satisfied certain conditions:

a. It must be a solution of Laplace's equation.
b. It must not be infinite everywhere on the polar axis.
c. It must lead to separability of the Hamilton-Jacobi equation.

Conditions (b) and (c) imply axial symmetry, so that no explicit assumption was really necessary for that. It happened, however, that a further condition was used, but not explicitly stated:

d. The spheroidal coordinates are centered around the center of mass and around the polar axis.

3. Transformations of the Rectangular Coordinate System

Giving up this fourth, unstated, condition permits an extension of the results. To see how, note that the Hamiltonian can be expressed as

$$H=\frac{1}{2}\frac{dx^2+dy^2+dz^2}{dt^2}+V(\xi,\eta) \tag{9}$$

and Laplace's equation as

$$\nabla^2 V\equiv\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V=0. \tag{10}$$

Postulating that $V(\xi,\eta)$ be a potential that satisfies (10) and leads to separability, we then ask what changes one can make in the rectangular coordinate system on which ξ, η, φ are based, without losing these properties. Since invariance of $ds^2=dx^2+dy^2+dz^2$ leads to invariance of ∇^2, the answer comes easily: any metric-preserving transformation. Actually the properties are preserved even if ds^2 and ∇^2 change by constant factors (conformal transformation). If, however, we write eqs (4) in the form

$$x+iy=c[(\xi^2+\eta^2)(1-\eta^2)]^{1/2}\exp i\varphi \tag{11.1}$$

$$z=c\xi\eta, \tag{11.2}$$

it is clear that we already have a disposable scale factor, viz, c, so that we cannot gain any added advantage by such a generalization. Also we do not gain anything by considering reflections, since these do not introduce any extra parameters that might be used to fit more J's. Hence our practical result: to obtain the greatest generalization of the results, we need consider only translations and rotations of the rectangular coordinate system.

Now from V1959a we have the result that

$$V=b_0\,\mathrm{Re}\,(\xi+i\eta)^{-1}+b_1\,\mathrm{Im}\,(\xi+i\eta)^{-1} \tag{12}$$

is the most general potential, based on the polar axis as z-axis and on the center of mass as origin for the spheroidal coordinates, that satisfies conditions a, b, and c. We may now translate and rotate the associated rectangular system and (12) will still have the desired properties a, b, and c. Since a translation is described by three parameters and a rotation by three parameters, it would appear at first that we could fit six more potential coefficients. Actually the number is only five, since of the three rotation parameters, it takes only two to fix the new z-axis. Since the third is used to fix the rotation of the new x and y axes around the latter, it is immaterial with a potential like (12), which is symmetric around the new z-axis.

4. Rotation

We assume that the polar axis is not an axis of symmetry, but that there exists such an axis OZ in the planet, with angular coordinates ψ and β. A plane through O, perpendicular to OZ, intersects the plane Oxy in the line of nodes N, which we take as the new X-axis. Then if $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along Ox, Oy, and Oz, unit vectors $\mathbf{k}'$ and $\mathbf{i}'$ along OZ and OX are given by

$$\mathbf{k}' = \mathbf{i} \sin\psi \cos\beta + \mathbf{j} \sin\psi \sin\beta + \mathbf{k} \cos\psi \tag{13.1}$$

$$\mathbf{i}' \sin\psi = \mathbf{k}' \times \mathbf{k} = \mathbf{i} \sin\psi \sin\beta - \mathbf{j} \sin\psi \cos\beta, \tag{13.2}$$

so that

$$\mathbf{i}' = \mathbf{i} \sin\beta - \mathbf{j} \cos\beta \tag{13.3}$$

and

$$\mathbf{j}' = \mathbf{k}' \times \mathbf{i} = \mathbf{i} \cos\psi \cos\beta + \mathbf{j} \cos\psi \sin\beta - \mathbf{k} \sin\psi. \tag{13.4}$$

Then since

$$\mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{i}'X + \mathbf{j}'Y + \mathbf{k}'Z, \tag{14}$$

we obtain the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{15.1}$$

where A is the square rotation matrix

$$A = \begin{pmatrix} \sin\beta & \cos\psi\cos\beta & \sin\psi\cos\beta \\ -\cos\beta & \cos\psi\sin\beta & \sin\psi\sin\beta \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}. \tag{15.2}$$

Here OZ is along the axis of symmetry of $V = b_0 \mathrm{Re}(\xi + i\eta)^{-1} + b_1 \mathrm{Im}(\xi + i\eta)^{-1}$, having angular coordinates ψ and β, the axis OX is along the line of nodes, and the sense of OY is given by the right-hand screw rule.

5. Translation Plus Rotation

If we also translate the origin from O to O' by a vector

$$\mathbf{OO}' \equiv \mathbf{s} = \mathbf{i}s_1 + \mathbf{j}s_2 + \mathbf{k}s_3 \tag{16}$$

we find

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{17}$$

and

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A^{-1}\begin{pmatrix} x-s_1 \\ y-s_2 \\ z-s_3 \end{pmatrix} = \begin{pmatrix} \sin\beta & -\cos\beta & 0 \\ \cos\psi\cos\beta & \cos\psi\sin\beta & -\sin\psi \\ \sin\psi\cos\beta & \sin\psi\sin\beta & \cos\psi \end{pmatrix}\begin{pmatrix} x-s_1 \\ y-s_2 \\ z-s_3 \end{pmatrix}. \tag{17.1}$$

Then

$$Z = (x-s_1)\sin\psi\cos\beta + (y-s_2)\sin\psi\sin\beta + (z-s_3)\cos\psi = (\mathbf{r}-\mathbf{s})\cdot\mathbf{k}' \tag{18}$$

Also, if $\mathbf{R}$ is the position vector $\mathbf{i}'X+\mathbf{j}'Y+\mathbf{k}'Z$, then $\mathbf{R}=\mathbf{r}-\mathbf{s}$ and

$$R^2 = r^2 + s^2 - 2\mathbf{s}\cdot\mathbf{r}. \tag{19}$$

Our separating potential is then

$$V = b_0 \mathrm{Re}\,(\xi+i\eta)^{-1} + b_1\,\mathrm{Im}(\xi+i\eta)^{-1}, \tag{20}$$

where

$$X+iY = c[(\xi^2+1)\,(1-\eta^2)]^{1/2}\exp i\varphi', \tag{21.1}$$

$$Z = c\xi\eta \tag{21.2}$$

and φ' is the φ-angle relative to $O'XYZ$.

6. Spherical Harmonic Expansion Relative to the Original System $Oxyz$

In order to fit the coefficients of potential, we must now expand the potential $V(\xi, \eta)$ in spherical harmonics relative to the original system $Oxyz$. Since we shall now obtain tesseral harmonics also, we have then to compare the expansion to be obtained with

$$V = -\frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n J_n P_n(\cos\theta) + \sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=1}^{n} P_n^m(\cos\theta)(C_{n,m}\cos m\varphi + S_{n,m}\sin m\varphi)\right]. \tag{22}$$

In this section only, we use θ to mean colatitude, rather than latitude, of a field point relative to $Oxyz$. This is to facilitate use of the addition theorem for spherical harmonics and because ψ is the colatitude of $O'Z$.

To obtain the expansion of $V(\xi, \eta)$ in (20), we need to expand $(\xi+i\eta)^{-1}$. Now

$$(\xi+i\eta)^2 = \xi^2 - \eta^2 + 2i\xi\eta \tag{23}$$

and from eqs (21)

$$\xi^2 - \eta^2 = \frac{R^2}{c^2} - 1 \tag{24.1}$$

$$2i\xi\eta = 2i\frac{Z}{c}. \tag{24.2}$$

From (23), (24), (18), and (19) it then follows, if $\mathbf{s}^2 = s^2$, that

$$(\xi+i\eta)^2 = \frac{r^2}{c^2}\left[1 + \frac{s^2-c^2}{r^2} - 2i\frac{c}{r}\mathbf{s}\cdot\mathbf{k}' + \frac{2}{r^2}\mathbf{r}\cdot(ic\mathbf{k}' - \mathbf{s}\right]. \tag{24}$$

But

$$(ic\mathbf{k}'-\mathbf{s})^2=s^2-2ic\,\mathbf{s}\cdot\mathbf{k}'-c^2, \tag{25}$$

since $\mathbf{k}'^2=1$, so that

$$(\xi+i\eta)^2=\frac{r^2}{c^2}\left[1+\frac{(ic\mathbf{k}'-\mathbf{s})^2}{r^2}+\frac{2}{r}(ic\mathbf{k}'-\mathbf{s})\cdot\mathbf{1}_r\right], \tag{26}$$

where $\mathbf{1}_r$ is a unit vector along $\mathbf{r}$. Then

$$(\xi+i\eta)^{-1}=\frac{c}{r}\left[1+\frac{2}{r}(ic\mathbf{k}'-\mathbf{s})\cdot\mathbf{1}_r+\left(\frac{ic\mathbf{k}'-\mathbf{s}}{r}\right)^2\right]^{-1/2}. \tag{27}$$

Since c and $|\mathbf{s}|$ will both be small compared to r, we have

$$\left|\frac{2}{r}(ic\mathbf{k}'-\mathbf{s})\cdot\mathbf{1}_r+\left(\frac{ic\mathbf{k}'-\mathbf{s}}{r}\right)^2\right|\leqslant 1 \tag{28}$$

and we then have the valid binomial expansion

$$(\xi+i\eta)^{-1}=\frac{c}{r}\left[1-\frac{1}{r}(ic\mathbf{k}'-\mathbf{s})\cdot\mathbf{1}_r-\frac{1}{2}\left(\frac{ic\mathbf{k}'-\mathbf{s}}{r}\right)^2+\ldots\right]. \tag{29}$$

Now the term $-cr^{-2}(ic\mathbf{k}'-\mathbf{s})\cdot\mathbf{1}_r$ will give rise to first harmonic terms unless we choose $\mathbf{s}$ properly. From (20) we find that the choice

$$b_0\mathbf{s}-b_1\,c\mathbf{k}'=0 \tag{30}$$

will eliminate terms in r^{-2} from $V(\xi,\eta)$ and thus place the origin O at the center of mass. Thus we must choose $\mathbf{s}$ to be either parallel or antiparallel to $\mathbf{k}'$ (implying that the new Z-axis must pass through the old origin, even after the combined translation plus rotation), so that

$$\mathbf{s}=s\mathbf{k}' \qquad (31.1), \qquad b_0s-b_1c=0, \tag{31.2}$$

where the sign of s is still undetermined. Then

$$ic\mathbf{k}'-\mathbf{s}=(ic-s)\mathbf{k}' \qquad (32.1), \qquad (ic\mathbf{k}'-\mathbf{s})^2=(ic-s)^2 \tag{32.2}$$

Insertion of (32) into (27) then gives

$$(\xi+i\eta)^{-1}=\frac{c}{r}\left[1-\frac{2}{r}(s-ic)\mathbf{k}'\cdot\mathbf{1}_r+\left(\frac{s-ic}{r}\right)^2\right]^{-1/2}. \tag{33}$$

Since

$$\mathbf{l}_r=\mathbf{i}\,\sin\theta\cos\varphi+\mathbf{j}\,\sin\theta\sin\varphi+\mathbf{k}\,\cos\theta, \tag{34}$$

it follows from (34) and (13.1) that, if U is the angle between $\mathbf{k}'$ and $\mathbf{1}_r$,

$$\mathbf{k}'\cdot\mathbf{1}_r=\cos U=\cos\psi\cos\theta+\sin\psi\sin\theta\cos(\varphi-\beta). \tag{35}$$

Then, from (33) and (35),

$$(\xi+i\eta)^{-1}=\frac{c}{r}\left[1-\frac{2}{r}\,(s-ic)\,\cos U+\left(\frac{s-ic}{r}\right)^2\right]^{-1/2} \tag{36}$$

$$=\frac{c}{r}\sum_{n=0}^{\infty}\left(\frac{s-ic}{r}\right)^n P_n(\cos U). \tag{37}$$

Insertion of the addition theorem for spherical harmonics

$$P_n(\cos U)=P_n(\cos\psi)\,P_n(\cos\theta)+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\psi)\,P_n^m(\cos\theta)\,\cos(m\varphi-m\beta) \tag{38}$$

into (37) then yields

$$(\xi+i\eta)^{-1}=\frac{c}{r}\sum_{n=0}^{\infty}\left(\frac{s-ic}{r}\right)^n\left[P_n(\cos\psi)\,P_n(\cos\theta)\right.$$

$$\left.+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\psi)\,P_n^m(\cos\theta)\,\cos(m\varphi-m\beta)\right]. \tag{39}$$

With application of (20) and (31.2) to (39) we may readily verify that the terms involving J_1, $C_{1,1}$, *and* $S_{1,1}$ are missing from (22), so that we have indeed placed the origin at the center of mass. We began with eight disposable parameters b_0, b_1, c, s_1, s_2, s_3, ψ, and β and after eliminating the first harmonics we are left with five, viz, b_0, s, c, ψ, and β, b_1 being fixed by (31.2).

On comparing (39) and (20) with (22), we next find that to fit the zeroth harmonic, we need

$$b_0c=-\mu. \tag{40}$$

Then

$$b_0=-\frac{\mu}{c}\text{ and }b_1=-\frac{\mu s}{c^2}. \tag{41}$$

Let us now put

$$s=\sigma r_e\cos\gamma\qquad c=\sigma r_e\sin\gamma, \tag{42}$$

so that

$$s-ic=\sigma r_e\exp(-i\gamma),\qquad (s-ic)^n=\sigma^n r_e^n\exp(-in\gamma). \tag{43}$$

From (39), (20), and (22), we then find for the *zonal harmonics*,

$$\mu r_e^n J_n=cP_n(\cos\psi)\,[b_0\,\mathrm{Re}(s-ic)^n+b_1\,\mathrm{Im}(s-ic)^n]. \tag{44}$$

Application of (43) to (44) then gives

$$J_n=\sigma^n P_n(\cos\psi)\,\frac{\sin(n-1)\gamma}{\sin\gamma}, \tag{45}$$

yielding the expected results $J_0=-1$, $J_1=0$, and

$$J_2=\sigma^2\,P_2(\cos\psi) \tag{46.1}$$

$$J_3=2\sigma^3\,P_3(\cos\psi)\cos\gamma \tag{46.2}$$

$$J_4 = \sigma^4 P_4(\cos\psi)\frac{\sin 3\gamma}{\sin\gamma}. \tag{46.3}$$

For the *tesseral and sectorial harmonics* we find

$$C_{n,m} + iS_{n,m} = -2\frac{(n-m)!}{(n+m)!}\sigma^n P_n^m(\cos\psi)\frac{\sin(n-1)\gamma}{\sin\gamma}\exp(im\beta), \tag{47}$$

leading to the expected results $C_{1,1} = S_{1,1} = 0$ and to

$$C_{2,1} = -\frac{\sigma^2}{2}\sin 2\psi\cos\beta \tag{48.1}$$

$$S_{2,1} = -\frac{\sigma^2}{2}\sin 2\psi\sin\beta \tag{48.2}$$

$$C_{2,2} = -\frac{\sigma^2}{4}\sin^2\psi\cos 2\beta \tag{48.3}$$

$$S_{2,2} = -\frac{\sigma^2}{4}\sin^2\psi\sin 2\beta. \tag{48.4}$$

We now have four parameters remaining, σ, ψ, γ, and β, to be fitted to certain of the potential coefficients above. Clearly we must fit to J_2. The remaining choices are to fit to J_3 or J_4 and to $C_{2,1}$ and $S_{2,1}$ or to $C_{2,2}$ and $S_{2,2}$. Now it is clear that in any practical application ψ would have to be a very small angle. Let

$$J_{2,1} = (C_{2,1}^2 + S_{2,1}^2)^{1/2} \qquad J_{2,2} = (C_{2,2}^2 + S_{2,2}^2)^{1/2}. \tag{49}$$

From (48), $J_{2,2}$ would be of order $J_{2,1}^2$, making $J_{2,1}$ much larger than $J_{2,2}$. This is contrary to all experience, since for the earth $J_{2,2}$ is a measure of equatorial ellipticity and $J_{2,1}$ of the variation in latitude produced by wobbling of the polar axis. We should therefore fit not to $C_{2,2}$ and $S_{2,2}$ but to $C_{2,1}$ and $S_{2,1}$. The resulting much too small values for $C_{2,2}$ and $S_{2,2}$ simply mean that we cannot fit them by this model.

7. The Physical Significance of the Rotation ψ, β

As we have defined it, ψ is the angle between the polar axis and the axis of symmetry of the separating potential $V = b_0\,\mathrm{Re}(\xi + i\eta)^{-1} + b_1\,\mathrm{Im}(\xi + i\eta)^{-1}$. Now nonvanishing of ψ leads to nonvanishing of $J_{2,1}$, which would actually vanish if Oz were a principal axis of the primary body. It is thus natural to identify the axis of symmetry of $V(\xi, \eta)$ with that principal axis which is closest to the axis of rotation.

For the earth $\psi \approx 0.1''$, the mean variation of latitude associated with the wandering of the pole. For the moon it is likewise expected to be very small.

Furthermore, when we use a potential that has symmetry about some axis other than that of rotation, the potential, and thus the Hamiltonian, become explicit functions of time, if we use an inertial reference system. This circumstance would defeat the separability. There is another possibility, that of letting the system $Oxyz$ be rigidly attached to the primary body, as we have actually done above. We should then have to treat the effect of rotation on the orbit of a satellite by means of apparent forces. They are not explicit functions of time, but they also destroy the separability, unless they are so small that we could neglect them. There might be some point in such a procedure for the moon, since its rotation is relatively slow, but even for the moon the apparent forces will undoubtedly be a good deal larger than those arising from $J_{2,1}$. If we could have fitted $J_{2,2}$, the story might have been different.

All in all the rotational invariance of $V(\xi, \eta)$ does not appear to lead to any practical results. I have included a treatment of it partly to verify that we do not miss anything useful by abstaining from its use, but mostly because its physical significance is so transparently clear. When we make such a rotational transformation, the new axis that we find is an axis of symmetry of the resulting potential. This stimulates us to search for a physical meaning of the translation $\mathbf{s}$, which turns out to be of real practical value, but whose physical significance is much less apparent.

8. Results of the Translational Invariance of $V(\xi, \eta)$

On putting $\psi = 0$, we thereby eliminate the rotational transformation and obtain the results of the pure translational invariance. We then have left only two paramters, σ and γ, to be fitted to the J_n's. By (45) we thus obtain

$$J_n = \sigma^n \frac{\sin (n-1)\gamma}{\sin \gamma}, \tag{50}$$

leading to

$$J_2 = \sigma^2 \tag{51.1}$$

$$J_3 = 2\sigma^3 \cos \gamma \tag{51.2}$$

$$J_4 = \sigma^4(4 \cos^2 \gamma - 1) \tag{51.3}$$

$$J_5 = 4\sigma^5 \cos \gamma(2 \cos^2 \gamma - 1). \tag{51.4}$$

We must certainly fit to J_2. Thus

$$\sigma = J_2^{1/2}. \tag{52}$$

Since (51.3) and (51.5) lead to

$$4 \cos^2 \gamma = \frac{J_4 + J_2^2}{J_2^2}, \tag{52.1}$$

we could fit to J_4 only if $J_4 + J_2^2$ were positive. Although the various investigators still find widely differing values of J_4, they now agree that $J_4 + J_2^2$ is negative. Thus we cannot fit to J_4.

On fitting to J_3, we then find

$$\cos \gamma = \tfrac{1}{2} J_3 J_2^{-3/2}, \tag{53}$$

which leads by (42) and (52) to

$$s = \tfrac{1}{2} r_e \frac{J_3}{J_2} \tag{54.1}$$

$$c = r_e J_2^{1/2} \left(1 - \frac{1}{4} \frac{J_3^2}{J_2^3}\right)^{1/2}. \tag{54.2}$$

With $J_2 = (1.083)10^{-3}$ and $J_3 \approx -(2.4)10^{-6}$, we find $\frac{1}{4} J_3^2 J_2^{-3} \approx 4 \times 10^{-3}$, so that the new value for c is about 2 parts in a thousand smaller than the value arrived at in V1959a. With $\psi = 0$, we have $\mathbf{s} = s\mathbf{k}$, where $s \approx -7.09$ km. The values of J_4 and J_5 corresponding to the above values of σ and γ are

$$J_4 = -J_2^2 + \frac{J_3^2}{J_2} \tag{55}$$

$$J_5 = -2J_3\left(J_2 - \frac{1}{2}\frac{J_3^2}{J_2}\right). \tag{56}$$

The value of J_4 given by the model is thus little changed from the earlier value of $-J_2^2$ (V1959b) and the values of J_5 and all higher J's are negligibly small. The fit to the potential is thus exact through the third zonal harmonic and accounts for at least two-thirds of the fourth zonal harmonic.

9. The New Form for the Potential

From (20) and (41) and $\rho = c\xi$, we now have

$$V = -\frac{\mu(\rho - \eta s)}{\rho^2 + c^2\eta^2} \tag{57}$$

If we place

$$s = -\delta, \tag{58}$$

where

$$\delta = \tfrac{1}{2} r_e \frac{|J_3|}{J_2} \tag{59}$$

is the magnitude of the distance from the new origin of spheroidal coordinates to the origin, we obtain

$$V = -\mu \frac{\rho + \eta\delta}{\rho^2 + c^2\eta^2}. \tag{60}$$

This of the same form as eq (49) in V1959b and δ has the same meaning that it had there, viz, the distance northward from the origin of spheroidal coordinates to the center of mass. The difference is that in the earlier paper the origins of spheroidal coordinates and of spherical harmonics were always the same. Had I kept $\delta \neq 0$ but expanded the spheroidal potential in spherical harmonics about the center of mass, I would have been able to fit J_3 at that time. However, the more general expressions for the quartic $G(\eta)$ that appear in eq (63) of the earlier paper can now be used.

The connections between rectangular, spherical, and spheroidal coordinates are now of course a little different. They become

$$x + iy = r\cos\theta \exp i\varphi = [(\rho^2 + c^2)\ (1 - \eta^2)]^{1/2} \exp i\varphi \tag{61.1}$$

$$z = r\sin\theta = -\delta + \rho\eta \tag{61.2}$$

Actually the only difference is the appearance of δ in (61.2).

Previously, with $\delta = 0$, a plane through the origin of spheroidal coordinates, perpendicular to the polar axis, had two important properties. It passed through the center of mass and was a plane of symmetry of the potential. Now, with $\delta \neq 0$, by (60) and (61.2), such a plane has neither of these properties. In contrast with the rotational transformation, which was to an axis of symmetry of the potential, the translation $\mathbf{s} = -\mathbf{k}\delta$, although it has the merit of enabling us to fit J_3 also, seems difficult to understand physically. We shall find later that a solution for some of the possible orbits will give us some physical grasp of the translation.

10. Remarks About Resonance Denominators

Before solving for the orbits, it is desirable to discuss briefly the bearing that the present topic has on the singularities found in perturbation theories of satellite orbits. If I is the inclination, the resonance denominator $1-5\cos^2 I$ has been found, by Brouwer 1959, Garfinkel 1959, Kozai 1962, V1963, and others, to occur with numerators $J_4+J_2^2$ and J_5, but not with the numerator J_3. In other words they are absent if $J_4+J_2^2$ and the odd J's vanish. One could previously understand their absence in such a case, because it corresponds, by (2), to the author's earlier spheroidal potential (3), with $\delta=0$. Such a potential, leading to separability, is not expected to lead to such resonance denominators in a perturbation theory. E.g., the solution of V1961b, V1962, gives exact secular terms and periodic terms which can be worked out to any degree of accuracy, with everything remaining finite when $1-5\cos^2 I=0$. Thus Kozai 1962 was able to do a partial check of his results by seeing whether his resonance denominators all vanished for $J_3=J_5=\ldots=0$ and $J_4+J_2^2=0$.

We now have a more stringent criterion for checking the accuracy of long-periodic terms in perturbation theories. The resonance denominators should all disappear when the J's are given by (50), with $\sigma=J_2^{1/2}$ and $\cos\gamma=\frac{1}{2}J_3J_2^{-3/2}$. I have tried to check Kozai [1962] in this way, but find the going difficult. Despite the great complexity of that paper, it may be that he has not carried terms to a sufficiently high order to make the test applicable.

11. Remarks About the Translation Parameter δ

Since $\delta=\frac{1}{2}r_eJ_2^{-1}|J_3|\approx 7$ km for the earth, it is indeed a large quantity to be produced by the nonvanishing of J_3. The latter produces a maximum change in the geoid of only about 20 m, a quantity of the order $r_e|J_3|$. To seek for some physical significance of δ, I shall consider the effect that J_3 has on some particularly simple orbits, viz, orbits which would reduce to circular equatorial orbits and circular polar orbits for $V=-\mu r^{-1}$.

12. The Special Orbits

To find these special orbits, we can take over some results from V1959b and V1961a, without re-solving the whole problem. The generalized momenta p_ρ and p_η are given by

$$p_\rho=\frac{\rho^2+c^2\eta^2}{\rho^2+c^2}\dot{\rho}=\pm\frac{F(\rho)^{1/2}}{\rho^2+c^2} \tag{62.1}$$

$$p_\eta=\frac{\rho^2+c^2\eta^2}{1-\eta^2}\dot{\eta}=\pm\frac{G(\eta)^{1/2}}{1-\eta^2}, \tag{62.2}$$

where $F(\rho)$ is a quartic with physical zeros $\rho_2\geqq\rho_1>0$ and $G(\eta)$ is the quartic

$$G(\eta)=-\alpha_3^2+(1-\eta^2)(\alpha_2^2+2\mu\eta\delta+2\alpha_1c^2\eta^2) \tag{63.1}$$

$$=\alpha_2^2-\alpha_3^2+2\mu\eta\delta+(2\alpha_1c^2-\alpha_2^2)\eta^2-2\mu\eta^3\delta-2\alpha_1c^2\eta^4. \tag{63.2}$$

In eqs (63) α_1 is the energy, α_3 the z-component of angular momentum, and α_2 the separation parameter, reducing to the total angular momentum in the Keplerian case. The physical zeros of $G(\eta)$ are η_0 and η_1, with

$$1\geqq\eta_0\geqq\eta_1\geqq-1, \tag{64}$$

since $\eta^2\leqq 1$.

We introduce the definitions

$$a_0 \equiv -\frac{\mu}{2\alpha_1},\ e_0 \equiv \left(1+\frac{2\alpha_1\alpha_2^2}{\mu^2}\right)^{1/2},\ i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2),\ p_0 \equiv a_0(1-e_0^2)$$

$$a \equiv \tfrac{1}{2}(\rho_1+\rho_2),\ e \equiv \frac{\rho_2-\rho_1}{\rho_2+\rho_1},\ p \equiv a(1-e^2). \tag{65}$$

Then for an orbit with $e=0$, $F(\rho)$ remains zero, and ρ is constant, by (62.1):

$$\rho = a = p. \tag{66}$$

From (4.14) and (4.15) of V1961a, we find for an *equatorial orbit* that

$$a_0 p_0 = a^2 + 3c^2 + O(J_2^2) \tag{67.1}$$

$$a_0 = a\left(1+\frac{c^2}{a^2}\right) + O(J_2^2). \tag{67.2}$$

For a polar orbit, it is clear from (61.1) that η^2 reaches the value unity, so that

$$\eta_0 = +1 \qquad \eta_1 = -1. \tag{68}$$

From the following paper we also have for a polar orbit

$$e = e_0 \qquad a = a_0 \qquad p = p_0. \tag{69}$$

If $e=0$, then $e_0=0$, so that $a_0=p_0$ and

$$\rho = a = a_0 = p_0 = p. \tag{70}$$

13. An "Equatorial" Orbit

For the case $J_3=0$ and thus $\delta=0$, it follows from V1961a that there is a solution corresponding to $\eta_0=\eta_1=0$, so that η then remains zero and the orbit is confined to the equatorial plane $z=0$. For $J_3 \neq 0$ and thus $\delta \neq 0$, such an orbit does not exist: a satellite moving in the plane $z=0$ would be driven out of that plane by the force

$$F_3 = \nabla\left[\mu\frac{r_e^3 J_3}{r^4}P_3(\sin\theta)\right] \tag{71}$$

arising from the third harmonic.

When $\delta \neq 0$, we see from (61) that the only circular orbit that always lies in a plane perpendicular to the polar axis is given by

$$\rho = \text{constant} \qquad \eta = \text{constant}. \tag{72}$$

But ρ can be constant only if $\rho_1=\rho_2$ and then

$$\rho = a. \tag{73}$$

The coordinate η can be constant only if

$$\eta_0 = \eta_1 \equiv \eta'. \tag{74}$$

To find the z-coordinate of this plane we must therefore find η'. Now when $\eta_0=\eta_1=\eta'$, η' is a double zero of $G(\eta)$ and is therefore a solution of

$$\left(\frac{dG}{d\eta}\right)_{\eta'}=0. \tag{75}$$

Thus, by (63.2) and (75), η' is a solution of

$$2\mu\delta+2(2\alpha_1c^2-\alpha_2^2)\eta'-6\mu\eta'^2\delta-8\alpha_1c^2\eta'^3=0. \tag{76}$$

If $\delta=0$, we know that $\eta'=0$, so that we may now assume η' to be small. On using the notation in (65) and solving (76) by successive approximations, we then find

$$\eta'=\frac{\delta}{p_0+c^2/a_0}\left[1-\frac{3\delta^2}{p_0^2}\right]+O(J_2^4). \tag{77}$$

Then, by (61.2), (73), and (77), the z-coordinate of the plane of this "equatorial" orbit is

$$z'=-\delta+\frac{a\delta}{p_0+c^2/a_0}+O(J_2^4). \tag{78}$$

From (67.1) and (67.2), we now find

$$p_0+\frac{c^2}{a_0}=a+\frac{3c^2}{a^2}+O(J_2^2), \tag{79}$$

so that

$$z'=-\delta+\frac{\delta}{1+3c^2/a^2}+O(J_2^3) \tag{80}$$

and

$$z'=-\frac{3c^2/a^2}{1+3c^2/a^2}\,\delta+O(J_2^3) \tag{81}$$

$$=-\frac{3c^2}{a^2}\,\delta+O(J_2^3), \tag{82}$$

since $\delta\equiv\frac{1}{2}r_eJ_2^{-1}|J_3|$ is of order J_2. By (54.2) we now have $c^2=r_e^2J_2+O(J_2^2)$, so that finally

$$z'=\frac{3}{2}\frac{r_e^3}{a^2}J_3+O(J_2^3). \tag{83}$$

For close orbits this gives $z'\approx-22$ m, a reasonable figure that is of the same order as the maximum change in the geoid produced by J_3.

As a check on the above expression, one may apply the perturbation theory for J_3 given in V1963, as applied to an orbit for which $e=0$ and $\sin I=0$. Since these quantities appear in denominators of the expressions for the variations of the Delaunay variables, one has to carry out limiting processes. It turns out that the long-periodic variations give $\delta z=0$, but that the short periodic ones give $\delta z=\frac{3}{2}r_e^3a^{-2}J_3$, equal to the above z'. Thus the present result is in agreement with perturbation calculations.

The result can also be verified directly from the equation $\ddot{z} = -\partial V/\partial z$. Here

$$-V = \frac{\mu}{r} + \tfrac{1}{2}\mu r_e^2 J_2 \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) + \tfrac{1}{2}\mu r_e^3 J_3 \left(\frac{3z}{r^5} - \frac{5z^3}{r^7}\right) + \ldots \tag{84}$$

Then

$$\ddot{z} = -\frac{\mu z}{r^3} + \tfrac{1}{2}\mu r_e^2 J_2 \left(-\frac{9z}{r^5} + \frac{15z^3}{r^7}\right) + \tfrac{1}{2}\mu r_e^3 J_3 \left(\frac{3}{r^5} - \frac{30z^2}{r^7} + \frac{35z^4}{r^9}\right) + \ldots \tag{85}$$

For an orbit of constant z and constant radius a, we have $\ddot{z} = 0$ and

$$r^2 = a^2 + z^2 = a^2 \left(1 + \frac{z^2}{a^2}\right). \tag{86}$$

On neglecting powers of z higher than the first, we then find

$$z = \frac{3}{2} \frac{r_e^3 J_3}{a^2 + \dfrac{9}{2} r_e^2 J_2} - = \frac{3}{2} \frac{r_e^3}{a^2} J_3 + O(J_2^3), \tag{87}$$

as we found before.

14. A Polar Orbit

Since we cannot interpret the translation parameter δ as the displacement produced by J_3 of the simplest equatorial orbit, let us now try the simplest polar orbit.

We first consider the case $\delta = 0$. Then by eqs (61)

$$x^2 + y^2 = (\rho^2 + c^2)(1 - \eta^2) \qquad z = \rho\eta. \tag{88}$$

If $e = 0$, then $\rho = a$ and

$$x^2 + y^2 = (a^2 + c^2)\left(1 - \frac{z^2}{a^2}\right),$$

leading to

$$\frac{x^2 + y^2}{a^2 + c^2} + \frac{z^2}{a^2} = 1. \tag{89}$$

This is the equation of the coordinate spheroid $\rho = a$ and simply means that when $e = 0$, the satellite's motion is restricted to it. If the orbit is polar, it is then the intersection of a meridian plane with such a spheroid. It is thus an ellipse with semiminor axis a, semimajor axis $(a^2 + c^2)^{1/2}$, with center (not focus!) at the center of mass of the planet. It is not a Keplerian ellipse, since the law of areas does not hold; if R is the distance from a focus to the satellite, it could not be produced by a potential $-\mu R^{-1}$ acting on the latter. That my original spheroidal potential [V1959a] leads to the existence of such an orbit was first noticed by Aksenov, Grebenikov, and Demin [1964].

By (88), this elliptic polar orbit lies between the limits

$$z = \pm a. \tag{90}$$

If we now consider the case $\delta \neq 0$ and still choose $e=0$, then $\rho=a$ as before and by eqs (61) the orbit is the intersection of a meridian plane with the spheroid

$$\frac{x^2+y^2}{a^2+c^2}+\frac{(z+\delta)^2}{a^2}=1. \tag{91}$$

It is thus an ellipse unchanged in size or shape, but with z now varying between the limits

$$z_2=-\delta+a$$

$$z_1=-\delta-a, \tag{92}$$

by (61.2).

For $J_3=0$, the plane of symmetry of all such elliptic polar orbits was the plane $z=0$. For $J_3 \neq 0$, their plane of symmetry is now

$$z=-\delta=\tfrac{1}{2} r_e J_2^{-1} J_3. \tag{93}$$

By (65) and (70), the energy $\alpha_1=-\frac{1}{2}\mu/a_0$ and $a_0=a$, so that all such orbits of fixed a have the same energy. From the conservation of energy one can show that any such orbit can be produced by first placing the satellite in the plane of symmetry, at a distance r_0 from the polar axis, and then projecting it parallel to the polar axis with initial velocity

$$\dot{z}_0=(\mu/a)^{1/2}, \tag{94}$$

where

$$a^2=r_0^2-c^2. \tag{95}$$

It appears that we have now arrived at a sort of physical interpretation of the translation parameter δ. When we go from the separable model with $J_3=0$ to the improved one that fits J_3, the necessary translation of the origin of spheroidal coordinates is equal to the displacement of the plane of symmetry of the elliptic polar orbits.

A certain verification can be obtained from the perturbation theory of V1963, by placing $e=0$ and $\eta_0=1$ into the long-periodic terms there given. We find for the variation of z

$$\delta z=\tfrac{1}{2}\frac{J_3}{J_2} r_e \sin^2\psi-\frac{J_3}{J_2} r_e \sin l \sin g \cos\psi. \tag{96}$$

Thus at either pole, with $\sin\psi=\pm 1$,

$$\delta z=\tfrac{1}{2} r_e \frac{J_3}{J_2}, \tag{97}$$

corresponding exactly to the above displacement. Moreover short-periodic averaging gives

$$\overline{\delta z}=\frac{J_3}{4J_2} r_e+\frac{J_3}{2J_2} r_e \sin^2 g \tag{98}$$

and long-periodic averaging then gives

$$\overline{\delta z}=\tfrac{1}{2}\frac{J_3}{J_2} r_e, \tag{99}$$

the same displacement as above.

We may sum up the physical results as follows. If we consider the equation $\ddot{\mathbf{r}} = -\nabla V$, where V includes only even zonal harmonics and where all J_k's higher than J_2 are much smaller, there must exist a class of polar orbits which are close to true ellipses, viz, those polar ellipses corresponding to $J_4 = -J_2^2$, $J_6 = J_2^3$, etc. All these true ellipses of a given energy generate a spheroid, and varying the energy furnishes the whole set of coordinate spheroids used in the spheroidal method. The equatorial plane is the plane of symmetry of these spheroids. It is also a plane through the center of mass and the plane of symmetry of the potential when only even zonal harmonics are included.

When we include J_3 also, if all higher zonal harmonics are small, there must again exist polar orbits which are very close to true ellipses, viz, those polar ellipses corresponding to the special choices made for the higher harmonics in the present paper. Again these new ellipses generate the new coordinate spheroids. The plane through $z = -\delta$, perpendicular to the polar axis, now does not pass through the center of mass and is now not a plane of symmetry of the potential, but it is again the plane of symmetry of the elliptical polar orbits.

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(Paper 70B1–165)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

Inclusion of the Third Zonal Harmonic in an Accurate Reference Orbit of an Artificial Satellite*

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(November 17, 1965)

The previous paper gave the general theory and physical principles involved in so modifying the author's spheroidal potential for an oblate planet as to permit exact inclusion of the effects of the third zonal harmonic of the planet's gravitational field. The present paper carries out the computational details necessary to derive the resulting orbit, which now corresponds to a potential fitted exactly through the third zonal harmonic and to about two-thirds of the fourth.

The accuracy of the orbit itself, as a solution for the given potential, depends on the accuracy of solution of a certain cubic equation. The paper works out this solution through terms of the third order in J_3, the coefficient of the second zonal harmonic, but its accuracy, and thus that of the secular terms, may be increased at will. Periodic terms are carried through the second order, but their accuracy may also be increased.

An obvious advantage of accounting for J_3 in this way is the absence of small denominators in e or $\sin I$ that occur in a perturbation theory. Another is the resulting increase in accuracy, through terms in J_2^2, of the long-periodic third harmonic terms.

Key Words: Third zonal harmonic, increased accuracy, spheroidal reference orbit, artificial satellite.

1. Introduction

The previous paper [2] [V1966] developed the theory and physical principles for inclusion of the third zonal harmonic of a planet's gravitational potential in an accurate reference orbit of an artificial satellite. The aim of the present paper is to work out the resulting generalized reference orbit in such detail as to permit practical calculation.

To begin, let ρ, η, φ be oblate spheroidal coordinates, satisfying the equations

$$x+iy=r\cos\theta\exp i\varphi=[(\rho^2+c^2)(1-\eta^2)]^{1/2}\exp i\varphi \tag{1.1}$$

$$z=r\sin\theta=-\delta+\rho\eta. \tag{1.2}$$

Here x, y, z are rectangular coordinates of a satellite in a Cartesian frame, with origin O at the center of mass of an oblate planet, Oz pointing along the polar axis, and Ox pointing toward its vernal equinox. Also r, θ, φ are the planetocentric distance, latitude, and right ascension.

If r_e is the equatorial radius and if

$$c_2=r_e^2J_2\left(1-\frac{1}{4}J_3^2J_2^{-3}\right) \tag{2}$$

$$\delta=-\tfrac{1}{2}r_eJ_2^{-1}J_3, \tag{3}$$

then the gravitational potential

$$V=-\mu(\rho^2+c^2\eta^2)^{-1}(\rho+\eta\delta) \tag{4}$$

leads to separability of the problem of satellite motion [V1966]. Here $\delta\approx 7$ km for the earth and

*Research supported by the National Aeronautical and Space Administration, Washington, D.C.

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[2] Since there will be so many references to my own papers, I abbreviate my name to the initial V in the body of the paper.

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the potential (4) leads to a fit of

$$V=-\frac{\mu}{r}\left[1-\sum_{2}^{\infty}\left(\frac{r_e}{r}\right)^n J_n P_n(\sin\theta)\right] \tag{5}$$

exactly through the third zonal harmonic. It also gives about two-thirds of the fourth harmonic and negligible values, of order J_2^3 or smaller, for the higher harmonics.

By V1959. if α_1 is the energy, α_3 the z-component of angular momentum, and α_2 the separation constant, the generalized momenta p_ρ, p_η, and p_φ are given by

$$p_\varphi=\alpha_3 \quad (6.1), \qquad p_\rho=\pm(\rho^2+c^2)^{-1}F(\rho)^{1/2} \quad (6.2), \qquad p_\eta=\pm(1-\eta^2)^{-1}G(\eta)^{1/2}. \tag{6.3}$$

Here $F(\rho)$ and $G(\eta)$ are the quartics

$$F(\rho)=c^2\alpha_3^2+(\rho^2+c^2)(-\alpha_2^2+2\mu\rho+2\alpha_1\rho^2) \tag{7.1}$$

$$G(\eta)=-\alpha_3^2+(1-\eta^2)(\alpha_2^2+2\mu\eta\delta+2\alpha_1c^2\eta^2). \tag{7.2}$$

The Hamilton-Jacobi function $W(\rho, \eta, \varphi)$ is then

$$W=\int p_\varphi d\varphi+\int p_\rho d\rho+\int p_\eta d\eta=\alpha_3\varphi+\int_{\rho_1}^{\rho}\pm(\rho^2+c^2)^{-1}F(\rho)^{1/2}d\rho+\int_{\eta_1}^{\eta}\pm(1-\eta^2)^{-1}G(\eta)^{1/2}d\eta. \tag{8}$$

The quartic $F(\rho)$ has two physical zeros, ρ_1 and ρ_2, and the quartic $G(\eta)$ has two physical zeros, η_1 and η_0. The possible motions are thus confined to the toroidal region

$$0<\rho_1\leqq\rho\leqq\rho_2 \tag{9.1}$$

$$-1\leqq\eta_1\leqq\eta\leqq\eta_0\leqq 1. \tag{9.2}$$

If β_1, β_2', and β_3' are constants, the orbit is then given by

$$t+\beta_1=\frac{\partial W}{\partial\alpha_1}, \qquad \beta_2'=\frac{\partial W}{\partial\alpha_2}, \qquad \beta_3'=\frac{\partial W}{\partial\alpha_3}. \tag{10}$$

With ρ_1 and η_1 as lower limits of the integrals in (8), these become

$$t+\beta_1=R_1+c^2N_1 \tag{11.1}$$

$$\beta_2'=-\alpha_2R_2+\alpha_2N_2 \tag{11.2}$$

$$\varphi=\beta_3'-c^2\alpha_3R_3+\alpha_3N_3. \tag{11.3}$$

Here the ρ-integrals are given by

$$R_1=\int_{\rho_1}^{\rho}\pm\rho^2F^{-1/2}d\rho, \qquad R_2=\int_{\rho_1}^{\rho}\pm F^{-1/2}d\rho, \qquad R_3=\int_{\rho_1}^{\rho}\pm(\rho^2+c^2)^{-1}F^{-1/2}d\rho \tag{12}$$

and the η-integrals by

$$N_1=\int_{\eta_1}^{\eta}\pm\eta^2G^{-1/2}d\eta \tag{12.1}$$

$$N_2=\int_{\eta_1}^{\eta}\pm G^{-1/2}d\eta \tag{12.2}$$

$$N_3 = \int_{\eta_1}^{\eta} \pm (1-\eta^2)^{-1} G^{-1/2} d\eta \cdot \tag{12.3}$$

With η_1 as lower limit in the η-integrals, rather than 0 as before, we shall find that in the Keplerian limit ($J_2 \to 0, J_3 \to 0$) the constant $-\beta_1$ will again reduce to the time of passage through perigee. The constant β_2', however, will not reduce to ω, the argument of perigee, but to $\omega + \pi/2$. Similarly β_3' will not reduce to Ω, the right ascension of the ascending node, but to $\Omega - \frac{1}{2}\pi \operatorname{sgn} \alpha_3$.

The ρ-integrals have already been worked out [V1961b]. The new η-integrals, however, will contain δ, involving J_3, and will thus have to be reevaluated. The task of the present paper is thus to reevaluate these η-integrals N_1, N_2, and N_3, insert them into (11), invert (11.1) and (11.2) to find ρ and η as functions of time, and then insert the results into (11.3) to find φ as a function of time. We shall then have the new reference orbit, corresponding to the generalized potential (4) that allows for exact inclusion of the third harmonic.

2. Factoring the Quartics

In order to evaluate the integrals, we first have to factor the quartics $F(\rho)$ and $G(\eta)$. The factoring of $F(\rho)$ was essentially done in V1961b and the only new task that we have in connection with it is to relate associated quantities to the new orbital elements, which will depend on the new factoring of $G(\eta)$. To factor $G(\eta)$ note that by (7.2)

$$G(\eta) = \alpha_2^2 - \alpha_3^2 + 2\mu\eta\delta + (2\alpha_1 c^2 - \alpha_2^2)\eta^2 - 2\mu\eta^3\delta - 2\alpha_1 c^2 \eta^4. \tag{13}$$

By (6.3) and (13), if the orbit ever reaches $\eta = 0$, we should have $\alpha_2^2 - \alpha_3^2 \geqq 0$. If indeed the odd harmonics are all put equal to zero, we should have $\delta = \frac{1}{2} r_e J_2^{-1} | J_3 | = 0$ and by (1.2) a motion $z = 0$ in the equatorial plane would be possible, with η remaining zero and $\alpha_2^2 - \alpha_3^2 \geqq 0$.

As soon as we insert the third harmonic, however, there arises a doubt that $\eta = 0$ will always be reached. Accordingly, we must leave the sign of $\alpha_2^2 - \alpha_3^2$ open. For bounded orbits we now have $\alpha_1 < 0$ and the table

η	$G(\eta)$
$-\infty$	$+\infty$
-1	$-\alpha_3^2 \leqq 0$
0	$\alpha_2^2 - \alpha_3^2$, sign either $\pm$
$+1$	$-\alpha_3^2 \leqq 0$
$+\infty$	$+\infty$

There are thus two non-physical zeros, between $-\infty$ and -1 and between $+1$ and $+\infty$, and two physical zeros, between -1 and $+1$. Call these latter η_1 and η_0, with $\eta_1 \leqq \eta_0$. If $\delta = 0$, $G(\eta)$ becomes a quadratic in η^2 and $\eta_1 = -\eta_0$, where $\eta_0 \geqq 0$, the equality sign holding when $\alpha_2^2 - \alpha_3^2 = 0$.

To see what happens when $\delta \neq 0$, make a plot of $G(\eta)$ versus η, noting that $G'(0) = 2\mu\delta > 0$. In order to avoid more than two zeros between $\eta = -1$ and $\eta = +1$, we must then have

if	that	and
$\alpha_2^2-\alpha_3^2>0$	$\eta_1<0<\eta_0$	$-\eta_0\eta_1>0$
$\alpha_2^2-\alpha_3^2=0$	$0=\eta_1<\eta_0$	$-\eta_0\eta_1=0$
$\alpha_2^2-\alpha_3^2<0$	$0<\eta_1\leqq\eta_0$	$-\eta_0\eta_1<0$

The paper V1961b indicated the following form for factoring $G(\eta)$:

$$G(\eta)=(\alpha_2^2-\alpha_3^2)\left(1-\frac{\eta}{\eta_0}\right)\left(1-\frac{\eta}{\eta_1}\right)(1+C_1\eta-C_2\eta^2). \tag{14}$$

Here, since $\eta_1\leqq\eta\leqq\eta_0$, we have

$$(\eta-\eta_1)(\eta_0-\eta)\geqq 0. \tag{15}$$

For $\alpha_2^2-\alpha_3^2>0$, we have $-\eta_0\eta_1>0$ and division gives

$$\left(1-\frac{\eta}{\eta_0}\right)\left(1-\frac{\eta}{\eta_1}\right)\geqq 0. \tag{16}$$

For $\alpha_2^2-\alpha_3^2<0$, we have $\eta_0\eta_1>0$ and division gives

$$\left(1-\frac{\eta}{\eta_0}\right)\left(1-\frac{\eta}{\eta_1}\right)<0. \tag{17}$$

In either case, since $G(\eta)\geqq 0$, we find that the factor $1+C_1\eta-C_2\eta^2>0$, where C_1 and C_2 will both be small compared to unity.

On comparing coefficients in (13) and (14), with

$$q_0\equiv\mu^{-1}(\alpha_2^2-\alpha_3^2) \tag{18}$$

$$P\equiv\tfrac{1}{2}(\eta_0+\eta_1) \tag{19}$$

$$S\equiv-\eta_0\eta_1, \tag{20}$$

and with the relations from V1961b, viz,

$$a_0\equiv-\tfrac{1}{2}\frac{\mu}{\alpha_1},\qquad e_0\equiv\left(1+\frac{2\alpha_1\alpha_2^2}{\mu^2}\right)^{1/2},\qquad p_0\equiv a_0(1-e_0^2) \tag{21.1}$$

$$\alpha_2^2=\mu p_0, \tag{21.2}$$

we have

$$2P+C_1S=\frac{2\delta}{q_0}S \tag{22.1}$$

$$1+C_2S-2C_1P=\frac{p_0+c^2/a_0}{q_0}S \tag{22.2}$$

$$C_1+2C_2P=\frac{2\delta}{q_0}S \tag{22.3}$$

$$C_2 = \frac{c^2}{a_0 q_0} S. \tag{22.4}$$

Along with three constants, to be derived later from β_1, β_2', and β_3', we shall choose as orbital elements

$$a = \tfrac{1}{2}(\rho_1 + \rho_2) \tag{23.1}$$

$$e = \frac{\rho_2 - \rho_1}{\rho_2 + \rho!} \tag{23.2}$$

$$p \equiv a(1 - e^2) \tag{23.3}$$

as in V1961b, and

$$S \equiv -\eta_0 \eta_1. \tag{23.4}$$

If $\delta = 0$, η_1 becomes $-\eta_0$, and S reduces to the old $\eta_0^2 = \sin^2 I$ of V1961b. Note that now, however, the new orbital element S can sometimes become negative, viz, for orbits sufficiently close to equatorial. Note that for a polar orbit $\eta_0 = +1$ and $\eta_1 = -1$, so that S then equals unity. In any case, since $-1 \leqq \eta_1 \leqq \eta_0 \leqq 1$, we have

$$S^2 \leqq 1. \tag{24}$$

Our next task is to express the factorization of the quartics $F(\rho)$ and $G(\eta)$ completely in terms of the elements a, e, and the new S. To do so, we must first express C_1, C_2, and P in terms of c, δ, a_0, p_0, and S; that is, we must eliminate q_0.

On inserting (22.4) into (22.3) and solving the resulting equation simultaneously with (22.1), we find

$$C_1 = \left(1 - \frac{c^2}{a_0 q_0} S^2\right)^{-1} \frac{2\delta}{q_0} S \left(1 - \frac{c^2}{a_0 q_0} S\right) \tag{25.1}$$

$$P = \left(1 - \frac{c^2}{a_0 q_0} S^2\right)^{-1} \frac{\delta}{q_0} S(1 - S). \tag{25.2}$$

If we now insert (25.1), (25.2), and (22.4) into (22.2) and call

$$\frac{p_0}{q_0} S \equiv u, \tag{26}$$

we find

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 p_0}(1 - S) + \frac{\left(\frac{2\delta}{p_0}\right)^2 (1 - S)\left(\frac{1}{u} - \frac{c^2}{a_0 p_0}\right)}{\left(\frac{1}{u} - \frac{c^2}{a_0 p_0} S\right)^2}. \tag{27}$$

For $\delta = 0$, S becomes $\sin^2 I$, u becomes $\sin^2 I / \sin^2 i_0$, where

$$i_0 \equiv \cos^{-1} \frac{\alpha_3}{\alpha_2}, \tag{28}$$

from V1961b, and (27) reduces to

$$\sin^2 i_0 = \sin^2 I + \frac{c^2}{a_0 p_0} \sin^2 I \cos^2 I, \tag{29}$$

in agreement with (4.8) of V1961b. Equation (27) is a cubic in u^{-1}, but we may solve it by successive approximations. To do so, note first that

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 p_0}(1 - S) + O(\delta^2). \tag{30}$$

If we insert (30) into the right side of (27), the error will be of order δ^4, i.e., of order J_2^4. Thus

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 p_0}(1 - S) + \epsilon, \tag{31.1}$$

where

$$\epsilon = \frac{\left(\frac{2\delta}{p_0}\right)^2 (1 - S)\left(1 - \frac{c^2}{a_0 p_0} S\right)}{\left[1 + \frac{c^2}{a_0 p_0}(1 - 2S)\right]^2} + O(J_2^4). \tag{31.2}$$

(Insertion of (31) in the right side of (27) would reduce the error to $O(J_2^6)$, but such accuracy is not needed.) Equation (31) frees u from q_0 and thus also C_1, C_2, and P, since by (22.4), (25), and (26)

$$C_2 = \frac{c^2}{a_0 p_0} u \tag{32.1}$$

$$C_1 = \left(1 - \frac{c^2}{a_0 p_0} S u\right)^{-1} \frac{2\delta}{p_0} u \left(1 - \frac{c^2}{a_0 p_0} u\right) \tag{32.2}$$

$$P = \left(1 - \frac{c^2}{a_0 p_0} S u\right)^{-1} \frac{\delta}{p_0} u (1 - S). \tag{32.3}$$

These quantities C_1, C_2, and P are then correct through order J_2^4.

Since $|S| \leqq 1$, we find from (31) that $\epsilon \geqq 0$ and $u^{-1} \geqq 1$. Thus

$$0 < u \leqq 1 \tag{33.1}$$

and then, by (32)

$$C_2 > 0, \qquad C_1 > 0, \qquad P \geqq 0. \tag{33.2}$$

In order to obtain all the required quantities as functions of a, e, and S, we must now summarize the factoring of $F(\rho)$, given in V1961b. We have

$$F(\rho) = -2\alpha_1(\rho - \rho_1)(\rho_2 - \rho)(\rho^2 + A\rho + B) \tag{34.1}$$

$$a \equiv \tfrac{1}{2}(\rho_1 + \rho_2), \qquad e \equiv \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \qquad p \equiv a(1 - e^2). \tag{34.2}$$

Then, with q_0 defined by (18) and a_0 and p_0 by (21), comparison of coefficients in (34.1) and (7.1) gives

$$2a - A = 2a_0 \tag{35.1}$$

$$B + ap - 2Aa = c^2 + a_0 p_0 \tag{35.2}$$

$$2aB - Aap = 2a_0 c^2 \tag{35.3}$$

$$Bap = a_0 p_0 c^2 q_0 / p_0. \tag{35.4}$$

We now regard a, p, S, c, and δ as known and have to solve the eight equations in (35), (32), and (26) for the eight unknowns a_0, p_0, q_0, A, B, C_1, C_2, and P. Here u is an auxiliary parameter given by (31).

From (35.4), (35.2), (26), and (31), we find

$$Bap = c^2 S\left[B + ap - 2Aa - c^2 S + \frac{4\frac{a_0}{p_0}\delta^2(1-S)\left(1 - \frac{c^2}{a_0 p_0}S\right)}{\left[1 + \frac{c^2}{a_0 p_0}(1-2S)\right]^2} + O(J_2^4)\right]. \tag{36}$$

Now since A and B are both of order J_2, we have from (35.1) and (35.2)

$$\frac{c^2}{a_0 p_0} = \frac{c^2}{ap} + O(J_2^2) \tag{37.1}$$

$$\frac{a_0}{p_0} = \frac{a}{p}\left(1 + \frac{c^2}{ap} + \frac{2A}{p} - \frac{A}{a} - \frac{B}{ap}\right) + O(J_2^3) \tag{37.2}$$

Because of the factor δ^2, of order J_2^2, insertion of (37) into (36) will not change the order of the error. Thus

$$Bap = c^2 S\left[B + ap - 2Aa - c^2 S + \frac{4a}{p}(1-S)\delta^2\left\{1 + \frac{c^2}{ap}(3S-1) + A\left(\frac{2}{p} - \frac{1}{a}\right) - \frac{B}{ap}\right\}\right] + O(J_2^3). \tag{38}$$

This is an equation involving only A, B, and "known" quantities. Another follows by combination of (35.1) and (35.3). It is

$$2aB - Aap = (2a - A)c^2. \tag{39}$$

The solution of (38) and (39) for A and B is

$$A = \frac{-2ac^2(ap - c^2S)(1-S) + \frac{8a^2c^2}{p}\delta^2\left\{1 + \frac{c^2}{ap}(3S-2)\right\}S(1-S)}{(ap - c^2)(ap - c^2S) + 4a^2c^2S + \frac{4c^2}{p^2}\delta^2(3ap - 4a^2 - c^2)S(1-S)} \tag{40.1}$$

$$B = (2a)^{-1}(ap - c^2)A + c^2. \tag{40.2}$$

For a polar orbit, for which $S=1$, we have from (40) and (35)

$$A=0 \qquad B=c^2 \qquad a_0=a \qquad p_0=p. \tag{41}$$

For such a polar orbit, if $e=0$, we then have

$$\rho=a=a_0=p=p_0, \qquad e_0=0. \tag{42}$$

Equations (40) give us A and B in terms of known quantities. To find the others, note that a_0 then follows from (35.1), p_0 from (35.2), q_0 from (26) and (31), and C_2, C_1, and P from (32). With use of these equations, the errors arising from the approximate solution of the cubic (27) are of order J_2^5 for all the unknowns except q_0. For q_0 the error is of order J_2^4. The "known" quantities are c, δ, a, p, and S. Here $c=r_e J_2^{1/2}\left(1-\frac{1}{4}J_3^2 J_2^{-3}\right)^{1/2}$, $\delta=\frac{1}{2}r_e J_2^{-1}|J_3|$, a is a constant "semi-major axis," p a constant "semi-latus rectum," and S a constant quantity analogous to the squared sine of the inclination for inclinations not too close to zero. For inclinations close to zero, however, S may be negative for $\delta \neq 0$.

3. The η-Integrals

From (14), (18), (19), and (20), we find

$$G(\eta)=\mu q_0 S^{-1}(S+2P\eta-\eta^2)(1+C_1\eta-C_2\eta^2). \tag{43}$$

Now

$$S+2P\eta-\eta^2 \equiv P^2+S-(\eta-P)^2. \tag{44}$$

The physical zeros η_1 and η_0 of $G(\eta)$ are the zeros of (44), viz,

$$\eta_0=P+(P^2+S)^{1/2} \tag{45.1}$$

$$\eta_1=P-(P^2+S)^{1/2}. \tag{45.2}$$

For $\delta \neq 0$, S may be negative, with smallest possible value $S=-P^2$, corresponding to

$$\eta_0=\eta_1=\eta'=\frac{\delta}{p_0+c^2/a_0}\left(1-\frac{3\delta^2}{p_0^2}\right)+O(J_2^4), \tag{46}$$

from V1966.

With

$$Q \equiv (P^2+S)^{1/2} \geqq 0, \tag{47}$$

we may now introduce a variable ψ, such that

$$\eta=P+Q\sin\psi. \tag{48}$$

If ψ continually increases, so that $\dot{\psi}>0$ always, η will then vary between the limits η_1 and η_0 and the double sign in eqs (12) for the η-integrals will disappear, with ψ as integration variable. Thus ψ is a uniformizing variable.

Insertion of (48) into (43) then gives

$$G(\eta)=\mu q_0 Q^2 S^{-1}(1+C_1\eta-C_2\eta^2)\cos^2\psi. \tag{49}$$

and the factor $\pm G^{-1/2}d\eta$ in the integrals (12) becomes

$$\pm G^{-1/2}d\eta=\left(\frac{S}{\mu q_0 Q}\right)^{1/2}(1+C_1\eta-C_2\eta^2)^{-1/2}d\psi \tag{50.1}$$

$$=\alpha_2^{-1}u^{1/2}(1+C_1\eta-C_2\eta^2)^{-1/2}d\psi \tag{50.2}$$

by (26) and (21.2).

4. The Integral N_2

By (12.2), (50), and (48) we then find

$$\alpha_2 u^{-1/2}N_2=\int_{-\frac{\pi}{2}}^{\psi}(1+C_1\eta-C_2\eta^2)^{-1/2}d\psi. \tag{51}$$

Here C_1 and C_2 are both positive, of order J_2. To evaluate (51), we place

$$h\equiv iC_2^{1/2}\eta=O(J_2),\qquad \lambda\equiv\tfrac{1}{2}iC_1C_2^{-1/2}=O(J_2^{1/2}), \tag{52}$$

so that

$$1+C_1\eta-C_2\eta^2\equiv 1-2\lambda h+h^2. \tag{53}$$

Since $\lambda=O(J_2^{1/2})$, it follows that $|\lambda\pm\sqrt{\lambda^2-1}|=O(1)$ and

$$|h|\ll\text{smaller of }|\lambda\pm\sqrt{\lambda^2-1}|. \tag{54}$$

Thus a Legendre expansion is legitimate, so that

$$(1+C_1\eta-C_2\eta^2)^{-1/2}=(1-2\lambda h+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^nP_n(\lambda)=\sum_{n=0}^{\infty}i^nC_2^{n/2}\eta^nP_n(\lambda). \tag{55}$$

Then (51) and (55) lead to

$$\alpha_2 u^{-1/2}N_2=\sum_{n=0}^{\infty}F_n(C_1,C_2)\int_{-\frac{\pi}{2}}^{\psi}\eta^n d\psi, \tag{56.1}$$

where

$$F_n(C_1,C_2)\equiv i^nC_2^{n/2}P_n(\tfrac{1}{2}iC_1C_2^{-1/2}), \tag{56.2}$$

are real polynomials in C_1 and C_2. Through F_8 they are

$$F_0=1,\qquad F_1=-\frac{1}{2}C_1,\qquad F_2=\frac{3}{8}C_1^2+\frac{1}{2}C_2,\qquad F_3=-\frac{5}{16}C_1^3-\frac{3}{4}C_1C_2,$$

$$F_4=\frac{35}{128}C_1^4+\frac{15}{16}C_1^2C_2+\frac{3}{8}C_2^2,\qquad F_5=-\frac{63}{256}C_1^5-\frac{35}{32}C_1^3C_2-\frac{15}{16}C_1C_2^2,$$

$$F_6=\frac{231}{1024}C_1^6+\frac{315}{256}C_1^4C_2+\frac{105}{64}C_1^2C_2^2+\frac{5}{16}C_2^3,\qquad F_7=-\frac{429}{2048}C_1^7-\frac{693}{512}C_1^5C_2-\frac{315}{128}C_1^3C_2^2-\frac{35}{32}C_1C_2^3,$$

$$F_8=\frac{6435}{128}C_1^8+\frac{3003}{2048}C_1^6C_2+\frac{3465}{1024}C_1^4C_2^2+\frac{315}{128}C_1^2C_2^3+\frac{35}{128}C_2^4. \tag{57}$$

Higher F's are of order $J^{\frac{3}{2}}$ or smaller.

We now have to evaluate

$$\int_{-\frac{\pi}{2}}^{\psi}\eta^n d\psi=\int_{-\frac{\pi}{2}}^{\psi}(P-Q\sin x)^n dx=\int_0^w(P+Q\cos y)^n dy\equiv f_n(w), \tag{58.1}$$

where

$$w\equiv\pi/2+\psi. \tag{58.2}$$

As in V1961b, p. 178, we now readily show that

$$f_n(w)=\pi^{-1}wf_n(\pi)+g_n(w), \tag{59}$$

where $g_n(w)$ is an odd function of w, of period 2π. From that same paper

$$f_n(\pi)\equiv\int_0^{\pi}(P-Q\cos y)^n dy=\pi P^n(1-e_1^2)^{n/2}P_n[(1-e_1^2)^{-1/2}], \tag{60.1}$$

where $P_n(x)$ is a Legendre polynomial in x and where

$$e_1\equiv Q/P \tag{60.2}$$

$$0<e_1<1 \text{ for } S<0 \tag{60.3}$$

$$e_1>1 \text{ for } S>0. \tag{60.4}$$

For $S<0$, the factors $(1-e_1^2)^{n/2}$ and $P_n[(1-e_1^2)^{-1/2}]$ are both real, for all n. For $S>0$, they are both real if n is even and both imaginary if n is odd. In any case their product is real.

If we now introduce, as in V1961b, p. 178,

$$R_n(x)\equiv x^nP_n(1/x), \tag{61}$$

a polynomial of degree $[n/2]$ in x^2, we find

$$\pi^{-1}f_n(\pi)=G_n(P,Q) \tag{62.1}$$

$$G_n(P,Q)\equiv P^nR_n(\sqrt{1-e_1^2}), \tag{62.2}$$

polynomials of degree n in P and Q. Through $n=8$ they are given by

$$G_0=1,\ G_1=P,\ G_2=P^2+\frac{1}{2}Q^2,\ G_3=P^3+\frac{3}{2}PQ^2,$$

$$G_4 = P^4 + 3P^2Q^2 + \frac{3}{8}Q^4, \; G_5 = P^5 + 5P^3Q^2 + \frac{15}{8}PQ^4,$$

$$16\,G_6 = 16P^6 + 120\,P^4Q^2 + 90P^2Q^4 + 5Q^6, \qquad 16G_7 = 16P^7 + 168P^5Q^2 + 210P^3Q^4 + 35PQ^6$$

$$128\,G_8 = 128P^8 + 1972P^6Q^2 + 3360P^4Q^4 + 1120P^2Q^6 + 35Q^8. \tag{63}$$

On inserting (58), (59), and (62) into (56.1), we find

$$\alpha_2 u^{-1/2} N_2 = \left(\frac{\pi}{2} + \psi\right) B_2 + \sum_{n=0}^{\infty} F_n(C_1, C_2) \sum_{j=1}^{\infty} c_{nj} \cdot \sin jw, \tag{64.1}$$

where

$$B_2 = \sum_{n=0}^{\infty} F_n(C_1, C_2)\, G_n(P, Q). \tag{64.2}$$

Remembering that C_1, C_2, and P are all of order J_2 and that Q is of order J_2^0, we have from (64.2), (57), and (63), through order J_2^4:

$$\begin{aligned} B_2 = 1 &- \frac{1}{2}C_1P + \left(\frac{3}{8}C_1^2 + \frac{1}{2}C_2\right)(P^2 + \frac{1}{2}Q^2) + \frac{9}{64}C_2^2Q^4 - \frac{9}{8}C_1C_2PQ^2 \\ &- \frac{15}{32}C_1^3PQ^2 + \frac{45}{128}C_1^2C_2Q^4 + \frac{25}{256}C_2^3Q^6 + \frac{105}{1024}C_1^4Q^4 \\ &+ \frac{9}{8}C_2^2P^2Q^2 - \frac{225}{128}C_1C_2^2PQ^4 + \frac{525}{1024}C_1^2C_2^2Q^6 + \left(\frac{35}{128}\right)^2 C_2^4Q^8 + O(J_2^5). \end{aligned} \tag{65}$$

Note that B_2 contains $-\frac{1}{2}C_1P + \frac{3}{16}C_1^2Q^2$, proportional to $J_3^2/J_2^2 = O(J_2^2)$, so that inclusion of J_3 in the separable problem apparently leads to new second-order secular terms. We shall say more about this point later.

Let us check B_2 for $\delta = 0$, in which case it should reduce to

$$B_2 = \pi^{-1}\,K(q) = 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \frac{25}{256}q^6 + \left(\frac{35}{128}\right)^2 q^8 + \ldots, \tag{66}$$

given on p. 184 of V1961b. For $\delta = 0$, we have $C_1 = 0$, $P = 0$, $\eta_1 = -\eta_0$, $S = \eta_0^2 = \sin^2 I$, and $Q = \eta_0 = \sin I$. For C_2 we then find from (22.4)

$$C_2 = \frac{c^2}{a_0 p_0}\,\frac{p_0}{q_0}\,\eta_0^2 = \frac{c^2}{a_0 p_0}\,\frac{\alpha_2^2}{\alpha_2^2 - \alpha_3^2}\,\eta_0^2, \tag{67}$$

with use of (18) and (21.2). But in the notation of V1961b, $(\alpha_2^2 - \alpha_3^2)/\alpha_2^2 = \sin^2 i_0$, so that we find

$$C_2 = \frac{c^2}{a_0 p_0}\,\frac{\eta_0^2}{\sin^2 i_0} \tag{68.1}$$

$$= \eta_2^{-2}. \tag{68.2}$$

from p. 173 of V1961b. On inserting these values into (65) and using $q \equiv \eta_0/\eta_2$, we find

$$B_2 = 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \frac{25}{256}q^6 + \left(\frac{35}{128}\right)^2 q^8 + \ldots. \tag{69}$$

This agrees with (66), so that B_2 reduces correctly for $\delta = 0$.

We now have to find the periodic terms in (64.1), i.e., the coefficients c_{nj}. Since $F_5(c_1, c_2)$ is of order J_2^3, we need take n only through the value 4. By expressing $(P + Q\cos y)^k$, $k = 0, 1, 2, 3, 4$, as trigonometric polynomials, evaluating $\int_0^\omega (P - Q\cos y)^k dy$, picking out the periodic terms, and keeping results only through order J_2^3, we find from (56.1) and (57)

$$\alpha_2 u^{-1/2} N_2 = \left(\frac{\pi}{2} + \psi\right) B_2 + \left(P - \frac{3}{16} C_1 Q^2\right) C_2 Q \sin w$$

$$+ \frac{1}{32}\left[(4C_2 + 3C_1^2)Q^2 + 3C_2^2 Q^4\right] \sin 2w + \frac{C_1 C_2}{16} Q^3 \sin 3w$$

$$+ \frac{3}{256} C_2^2 Q^4 \sin 4w + \text{periodic terms of order } J_2^3. \tag{70}$$

In terms of $\psi = w - \pi/2$, this becomes

$$\alpha_2 u^{-1/2} N_2 = \left(\frac{\pi}{2} + \psi\right) B_2 + \left[C_2 Q\left(p + \frac{9}{16} C_1 Q_2\right) + \frac{1}{2} C_1 Q\right] \cos \psi$$

$$- \frac{1}{32}\left[(4C_2 + 3C_1^2)Q^2 + 3C_2^2 Q^4\right] \sin 2\psi - \frac{C_1 C_2}{16} Q^3 \cos 3\psi$$

$$+ \frac{3}{256} C_2^2 Q^4 \sin 4\psi + \text{periodic terms of } O(J_2^3). \tag{71}$$

For $\delta = 0$, we find that the periodic terms in (71) agree with those on p. 184 of V1961b, as they should.

5. The Integral N_1

By (12.1), (50), and (48), we have

$$\alpha_2 u^{-1/2} N_1 = \int_{-\frac{\pi}{2}}^{\psi} (1 + C_1\eta - C_2\eta^2)^{-1/2} \eta^2 d\psi. \tag{72}$$

Then, from (55), (56.2), and (58.1)

$$\alpha_2 u^{-1/2} N_1 = \sum_{n=0}^{\infty} F_n(C_1, C_2) f_{n+2}(w). \tag{73}$$

As before,

$$f_{n+2}(w) = wG_{n+2}(P, Q) + g_{n+2}(w), \tag{74}$$

where $g_{n+2}(w)$ is odd in w, with period 2π. Thus

$$\alpha_2 u^{-1/2} N_1 = \left(\frac{\pi}{2} + \psi\right) B_1' + \sum_{n=0}^{\infty} F_n(C_1, C_2) g_{n+2}(w), \tag{75.1}$$

where

$$B_1' \equiv \sum_{n=0}^{\infty} F_n(C_1, C_2)\, G_{n+2}(P, Q). \tag{75.2}$$

Since the integral N_1 has a factor $c^2 = O(J_2)$ in the kinetic equations (11), we need each $F_n G_{n+2}$ correct only through order J_2^3, in order to obtain secular terms correct through order J_2^4. From (75.2), (57), and (63), we then find

$$B_1' = \tfrac{1}{2} Q^2 + P^2 - \frac{3}{4} C_1 P Q^2 + \frac{3}{2} C_2 P^2 Q^2 + \frac{3}{64} (4C_2 + 3C_1^2) Q^4$$

$$- \frac{45}{32} C_1 C_2 P Q^4 + \frac{5}{256} (6C_2^2 + 15 C_1^2 C_2) Q^6 + \frac{175}{2048} C_2^3 Q^8 + O(J_2^4). \tag{76}$$

For $\delta = 0$, this reduces to

$$B_1' = \tfrac{1}{2} \eta_0^2 + \frac{3}{16} \eta_2^{-2} \eta_0^4 + \frac{15}{128} \eta_2^{-4} \eta_0^6 + \frac{175}{2048} \eta_2^{-6} \eta_0^8 + \ldots$$

or

$$B_1' \eta_0^{-2} = \tfrac{1}{2} + \frac{3}{16} q^2 + \frac{15}{128} q^4 + \frac{175}{2048} q^6 + \ldots \tag{77}$$

in agreement with the value of B_1 on p. 184 of V1961b.

To find the periodic terms in (75.1), we proceed just as for N_2, obtaining

$$\alpha_2 u^{-1/2} N_1 = \left(\frac{\pi}{2} + \psi\right) B_1' + \left(2PQ + \frac{3}{8} C_1 Q^3\right) \cos\psi$$

$$- \frac{1}{8} (2Q^2 + C_2 Q^4) \sin 2\psi - \frac{1}{24} C_1 Q^3 \cos 3\psi$$

$$+ \frac{1}{64} C_2 Q^4 \sin 4\psi + \text{periodic terms of } O(J_2^3). \tag{78}$$

For $\delta = 0$, the periodic terms agree with those appearing in (6.38) of V1961b.

6. The Integral N_3

By (12.3), (50), and (48), we have

$$\alpha_2 u^{-1/2} N_3 = \int_{-\frac{\pi}{2}}^{\psi} (1 - \eta^2)^{-1} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} d\psi. \tag{79}$$

Then, from (55) and (56.6),

$$\alpha_2 u^{-1/2} N_3 = \sum_{n=0}^{\infty} F_n(C_1, C_2) L_n \tag{80.1}$$

$$L_n \equiv \int_{-\frac{\pi}{2}}^{\psi} (1-\eta^2)\eta^n d\psi. \tag{80.2}$$

In particular

$$L_0 \equiv \int_{-\frac{\pi}{2}}^{\psi} (1-\eta^2)^{-1} d\psi \qquad (81.1) \qquad L_1 \equiv \int_{-\frac{\pi}{2}}^{\psi} (1-\eta^2)^{-1}\eta d\psi. \tag{81.2}$$

To evaluate the L_n's we first prove the lemmas

$$L_{2\sigma} = L_0 - \sum_{\tau=0}^{\sigma-1} \int_{-\frac{\pi}{2}}^{\psi} \eta^{2\tau} d\psi \qquad (\sigma \geqq 1) \tag{82.1}$$

$$L_{2\sigma+1} = L_1 - \sum_{\tau=0}^{\sigma-1} \int_{-\frac{\pi}{2}}^{\psi} \eta^{2\tau+1} d\psi \qquad (\sigma \geqq 1). \tag{82.2}$$

To prove them, note first that

$$\frac{\eta^n}{1-\eta} = \frac{1}{1-\eta} - \sum_{k=0}^{n-1} \eta^k \tag{83}$$

and

$$\frac{\eta^n}{1+\eta} = \frac{(-1)^n}{1+\eta} - \sum_{k=0}^{n-1} (-1)^{n+k} \eta^k. \tag{84}$$

Addition of (83) and (84) then gives

$$\frac{\eta^n}{1-\eta} + \frac{\eta^n}{1+\eta} = \frac{1}{1-\eta} + \frac{(-1)^n}{1+\eta} - \sum_{k=0}^{n-1} [1 + (-1)^{n+k}] \eta^k, \tag{85}$$

reducing to the two cases $n =$ odd and $n =$ even, viz.

$$\frac{\eta^{2\sigma}}{1-\eta^2} = \frac{1}{1-\eta^2} - \sum_{\tau=0}^{\sigma-1} \eta^{2\tau} \qquad (\sigma \geqq 1) \tag{86.1}$$

$$\frac{\eta^{2\sigma+1}}{1-\eta^2} = \frac{\eta}{1-\eta^2} - \sum_{\tau=0}^{\sigma-1} \eta^{2\tau+1} \qquad (\sigma \geqq 1). \tag{86.2}$$

The lemmas (82) then follow directly from (80.2) and (86).

With application of (58.1), (59), and (62), we then find from them

$$L_{2\sigma} = L_0 - w \sum_{\tau=0}^{\sigma-1} G_{2\tau}(P, Q) - \sum_{\tau=0}^{\sigma-1} g_{2\tau}(w), \qquad \sigma \geqq 1 \tag{87.1}$$

$$L_{2\sigma+1} = L_1 - w \sum_{\tau=0}^{\sigma-1} G_{2\tau+1}(P, Q) - \sum_{\tau=0}^{\sigma-1} g_{2\tau+1}(w), \quad \sigma \geqq 1, \tag{87.2}$$

where $w = \psi + \pi/2$. Insertion of (87) into (80.1) then gives

$$\alpha_2 u^{-1/2} N_3 = L_0 \sum_{\sigma=0}^{\infty} F_{2\sigma} + L_1 \sum_{\sigma=0}^{\infty} F_{2\sigma+1} + B_3 w$$

$$- \sum_{\sigma=1}^{\infty} F_{2\sigma} \sum_{\tau=0}^{\sigma-1} g_{2\tau}(w) - \sum_{\sigma=1}^{\infty} F_{2\sigma+1} \sum_{\tau=0}^{\sigma-1} g_{2\tau+1}(w), \tag{88.1}$$

$$B_3 \equiv -\sum_{\sigma=1}^{\infty} F_{2\sigma} \sum_{\tau=0}^{\sigma-1} G_{2\tau} - \sum_{\sigma=1}^{\infty} F_{2\sigma+1} \sum_{\tau=0}^{\sigma-1} G_{2\tau+1}. \tag{88.2}$$

To simplify the first terms in (88.1), note that by (56.2) and (52), if

$$h_0 \equiv iC_2^{1/2}, \tag{89}$$

the value of h for $\eta = 1$, we obtain

$$F_{2\sigma} = h_0^{2\sigma} P_{2\sigma}(\lambda) \qquad (90.1) \qquad F_{2\sigma+1} = h_0^{2\sigma+1} P_{2\sigma+1}(\lambda). \tag{90.2}$$

To evaluate the sums $\sum_0^{\infty} F_{2\sigma}$ and $\sum_0^{\infty} F_{2\sigma+1}$, note that

$$(1 - 2\lambda h_0 + h_0^2)^{-1/2} = \sum_0^{\infty} h_0^{2\sigma} P_{2\sigma}(\lambda) + \sum_0^{\infty} h_0^{2\sigma+1} P_{2\sigma+1}(\lambda) \tag{91.1}$$

and, since $P_{2\sigma}(-\lambda) = P_{2\sigma}(\lambda)$ and $P_{2\sigma+1}(-\lambda) = -P_{2\sigma+1}(\lambda)$, that

$$(1 + 2\lambda h_0 + h_0^2)^{-1/2} = \sum_0^{\infty} h_0^{2\sigma} P_{2\sigma}(\lambda) - \sum_0^{\infty} h_0^{2\sigma+1} P_{2\sigma+1}(\lambda). \tag{91.2}$$

Solving eqs (91) for each sum, with use of (90), we find

$$2\sum_0^{\infty} F_{2\sigma} = (1 - 2\lambda h_0 + h_0^2)^{-1/2} + (1 + 2\lambda h_0 + h_0^2)^{-1/2} \tag{92.1}$$

$$2\sum_0^{\infty} F_{2\sigma+1} = (1 - 2\lambda h_0 + h_0^2)^{-1/2} - (1 + 2\lambda h_0 + h_0^2)^{-1/2}. \tag{92.2}$$

Since $h = h_0$ when $\eta = 1$, we have from (53)

$$1 \mp 2\lambda h_0 + h_0^2 = 1 \pm C_1 - C_2. \tag{93}$$

Then from (92) and (93)

$$L_0 \sum_0^{\infty} F_{2\sigma} + L_1 \sum_0^{\infty} F_{2\sigma+1} = \frac{1}{2} L_0[(1 + C_1 - C_2)^{-1/2} + (1 - C_1 - C_2)^{-1/2}]$$

$$+ \frac{1}{2} L_1[(1 + C_1 - C_2)^{-1/2} - (1 - C_1 - C_2)^{-1/2}]. \tag{94}$$

For B_3, the coefficient of a secular term, we find from (88.2), (57), and (63), carrying terms through order J_2^4,

$$B_3 = -\frac{1}{2} C_2 - \frac{3}{8} C_1^2 - \left(\frac{35}{128} C_1^4 + \frac{15}{16} C_1^2 C_2 + \frac{3}{8} C_2^2\right)\left(1 + \frac{1}{2} Q^2\right)$$

$$- \frac{3}{8} C_2^2 P^2 - \left(\frac{105}{64} C_1^2 C_2^2 + \frac{5}{16} C_2^3\right)\left(1 + \frac{1}{2} Q^2 + \frac{3}{8} Q^4\right)$$

$$-\frac{35}{128}C_2^4\left(1+\frac{1}{2}Q^2+\frac{3}{8}Q^4+\frac{5}{16}Q^6\right)-\left(\frac{5}{16}C_1^3+\frac{3}{4}C_1C_2\right)P$$

$$+\frac{15}{16}C_1C_2^2\left(P+\frac{3}{2}PQ^2\right)+O(J_2^3). \qquad (95)$$

Here the term $-\frac{3}{8}C_1^2$ is proportional to $J_3^2/J_2^2=O(J_2^3)$, so that we apparently again find a second-order secular term produced by inclusion of J_3 in the separable problem.

To check (95) for $\delta=0$, put $C_1=P=0$, $C_2=\eta_2^{-2}$, and $Q=\eta_0$. It becomes

$$B_3=\frac{1}{2}\eta_2^{-2}+\frac{3}{8}\eta_2^{-4}\left(1+\frac{1}{2}\eta_0^2\right)+\frac{5}{16}\eta_2^{-6}\left(1+\frac{1}{2}\eta_0^2+\frac{3}{8}\eta_0^4\right)$$

$$+\frac{35}{128}\eta_2^{-8}\left(1+\frac{1}{2}\eta_0^2+\frac{3}{8}\eta_0^4+\frac{5}{16}\eta_0^6\right)+O(J_2^3), \qquad (95.1)$$

agreeing with (6.65) of V1961b through terms of order J_2^4.

To evaluate the periodic terms in (88.1) through order J_2^2, note that all coefficients F_n are of order J_2^3 or higher for $n\geqq 5$. Thus we need only g_0, g_1, and g_2, where g_n is the periodic part of

$$f_n\equiv\int_0^w(P-Q\cos y)^n dy.$$

Then

$$g_0=0,\ g_1=Q\sin w,\ g_2=-2PQ\sin w+\frac{Q^2}{4}\sin 2w. \qquad (96)$$

From (88.1) the relevant periodic terms are then $-F_3g_1$ and $-F_4g_2$. With the aid of (96) and (57), their sum then becomes through order J_2^2,

$$\text{Periodic terms}=-\frac{3}{4}C_1C_2Q\sin w-\frac{3}{32}C_2^2Q^2\sin 2w \qquad (97.1)$$

$$=-\frac{3}{4}C_1C_2Q\cos\psi+\frac{3}{32}C_2^2Q^2\sin 2\psi. \qquad (97.2)$$

We come next to the evaluation of L_0 and L_1. From (81), we have by expansion in partial fractions

$$2L_0=M_1'+M_2'$$

$$2L_1=M_1'-M_2', \qquad (98.1)$$

where

$$M_1'=\int_{-\pi/2}^{\psi}(1-\eta)^{-1}d\psi \qquad (99.1) \qquad M_2'=\int_{-\pi/2}^{\psi}(1+\eta)^{-1}d\psi. \qquad (99.2)$$

With the aid of (48) and the definitions

$$e_2\equiv\frac{Q}{1-P} \qquad (100.1) \qquad e_3\equiv\frac{Q}{1+P}, \qquad (100.2)$$

we find

$$M_1' = (1-P)^{-1} \int_{-\pi/2}^{\psi} (1 - e_2 \sin x)^{-1} dx \tag{101.1}$$

$$M_2' = (1+P)^{-1} \int_{-\pi/2}^{\psi} (1 + e_3 \sin x)^{-1} dx. \tag{101.2}$$

Here $e_2 \leqq 1$ and $e_3 \leqq 1$, as we shall now prove. Since $|S| \leqq 1$, we have by (31) that $u \approx 1$ and then by (32.3) that

$$P = \zeta(1-S) \qquad (102.1) \qquad \zeta = \left(1 - \frac{c^2}{a_0 p_0} Su\right)^{-1} \frac{\delta}{p_0} u \tag{102.2}$$

where $0 < \zeta < < 1$. Now by (47)

$$(1-P)^2 - Q^2 = 1 - 2P + P^2 - P^2 - S = 1 - 2P - S = (1-S)(1-2\zeta),$$

by (102). But $0 < \zeta < < 1$, so that $1 - 2\zeta > 0$ and $1 - S \geqq 0$, so that $(1-P)^2 - Q^2 \geqq 0$ and $1 - P \geqq Q$, since neither $1-P$ nor Q is negative. Thus $e_2 \leqq 1$. Also $(1+P)^2 - Q^2 = 1 + 2P - S \geqq 0$, since $P \geqq 0$ and $S \leqq 1$.
Thus $1 + P \geqq Q$ and $e_3 \leqq 1$.

If in (101.1) we place $x = y - \pi/2$ and in (101.2) we place $x = y + \pi/2$, we now find

$$M_1' = (1-P)^{-1} \int_0^{\psi+\pi/2} (1 + e_2 \cos y)^{-1} dy \tag{103.1}$$

$$M_2' = (1+P)^{-1} \int_{-\pi}^{\psi-\pi/2} (1 + e_3 \cos y)^{-1} dy. \tag{103.2}$$

To evaluate the M''s we define a function $E_e'(y)$ by

$$\cos E_e' = \frac{e + \cos y}{1 + e \cos y} \qquad \sin E_e' = \frac{(1-e^2)^{1/2} \sin y}{1 + e \cos y} \qquad (e < 1). \tag{104}$$

Then when $y = n\pi$, we find $\sin E_e' = 0$ and $\cos E_e' = (-1)^n$. We may therefore impose the further requirement that $E_e'(y) = y$ whenever $y = n\pi$. The relation between E_e' and y is then the same as that between the eccentric and true anomalies, respectively, of an ellipse with eccentricity e.

We next prove that

$$\int_0^w (1 + e \cos y)^{-1} dy = (1-e^2)^{-1/2} E_e'(w). \tag{105}$$

To prove it, we first derive from (104) the relations

$$\sin y = \frac{(1-e^2)^{1/2} \sin E'}{1 - e \cos E'} \tag{106}$$

$$\frac{1+e^2}{1+e\cos y} = 1 - e \cos E'. \tag{107}$$

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Differentiation of (107) gives

$$\frac{(1-e^2)\sin y\,dy}{(1+e\cos y)^2}=\sin E'dE'. \tag{108}$$

Division of (108) by (107) then shows that

$$\frac{\sin y\,dy}{1+e\cos y}=\frac{\sin E'dE'}{1-e\cos E'}=(1-e^2)^{-1/2}\sin y\,dE', \tag{109}$$

by (106). Thus

$$\frac{dy}{1+e\cos y}=(1-e^2)^{-1/2}\,dE', \tag{110}$$

from which theorem (105) follows at once, since $E'=0$ when $y=0$. Then, since $E'_e(0)=0$ and $E'_e(-\pi)=-\pi$, we find on applying (105) to (103),

$$M'_1=(1-P)^{-1}(1-e_2^2)^{-1/2}\,E'_2(\psi+\pi/2) \tag{111.1}$$

$$M'_2=(1+P)^{-1}(1-e_3^2)^{-1/2}\,[E'_3(\psi-\pi/2)+\pi]. \tag{111.2}$$

From (111), (100), and $Q^2=P^2+S$, it then follows that

$$M'_1=(1-S-2P)^{-1/2}E'_2(\psi+\pi/2) \tag{112.1}$$

$$M'_2=(1-S+2P)^{-1/2}[E'_3(\psi-\pi/2)+\pi]. \tag{112.2}$$

By (112), (98), and (102), we then find

$$(1-S)^{1/2}L_0=\frac{\pi/2}{\sqrt{1+2\zeta}}+\chi_0 \tag{113.1}$$

$$(1-S)^{1/2}L_1=\frac{-\pi/2}{\sqrt{1+2\zeta}}+\chi_1, \tag{113.2}$$

where

$$2\chi_0=\frac{E'_2(\psi+\pi/2)}{\sqrt{1-2\zeta}}+\frac{E'_3(\psi-\pi/2)}{\sqrt{1+2\zeta}} \tag{114.1}$$

$$2\chi_1=\frac{E'_2(\psi+\pi/2)}{\sqrt{1-2\zeta}}-\frac{E'_3(\psi-\pi/2)}{\sqrt{1+2\zeta}}. \tag{114.2}$$

From (88.1), (94), (97), and (113), there follows the complete expression for N_3

$$\begin{aligned}\alpha_2u^{-1/2}N_3=&\frac{1}{2}[(1+C_1-C_2)^{-1/2}+(1-C_1-C_2)^{-1/2}](1-S)^{-1/2}\left[\frac{\pi/2}{\sqrt{1+2\zeta}}+\chi_0\right]\\&+\frac{1}{2}[(1+C_1-C_2)^{-1/2}-(1-C_1-C_2)^{-1/2}](1-S)^{-1/2}\left[\frac{-\pi/2}{\sqrt{1+2\zeta}}+\chi_1\right]\\&+(\psi+\pi/2)B_3-\frac{3}{4}C_1C_2Q\cos\psi+\frac{3}{32}C_2^2Q^2\sin 2\psi+\text{periodic terms of }O(J_2^3).\end{aligned} \tag{115}$$

Here B_3 is given by (95) and χ_0 and χ_1 are given by (114).

To check this expression for $\delta=0$, note that then $C_1=0$, $\zeta=0$, $S=\eta_0^2$, $u=(\alpha_2^2-\alpha_3^2)^{-1}\ \alpha_2^2\eta_0^2$, so that $\alpha_2 u^{-1/2}$ becomes $(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}$. We have to compare with eq (6.64) of V1961b. Now we have already checked B_3 and the periodic terms in (115). The check will thus consist only in showing that the terms involving χ_0 and χ_1 in (115) reduce to the expression $(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\chi$ of V1961b.

Since $C_1=0$ for $\delta=0$, the term involving χ_1 disappears, because of the minus sign inside the brackets. Since C_2 reduces to η_2^{-2}, the first line becomes

$$(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\left(\frac{\pi}{2}+\chi_0\right).$$

Now the $\pi/2$ is an additive constant, arising only because we now have the lower limit η_1 in the η-integrals instead of zero. We shall later get rid of it on replacing the constant β_2' by another constant β_2. Thus we now have only to verify that χ_0 reduces to χ when $\delta=0$. When $\delta=0$, $e_2=(1-P)^{-1}Q$ reduces to η_0, and we find from (114.1) and (104), with $\zeta=0$, that

$$\cos 2\chi_0=\frac{\cos^2\psi-(1-\eta_0^2)\ \sin^2\psi}{1-\eta_0^2\sin^2\psi}\qquad \sin 2\chi_0=\frac{2\sqrt{1-\eta_0^2}\sin\psi\cos\psi}{1-\eta_0^2\sin^2\psi}.$$

But the χ of V1961b, by p. 516 of V1961c, satisfies

$$\cos\chi=\frac{\cos\psi}{\sqrt{1-\eta_0^2\sin^2\psi}}\qquad \sin\chi=\frac{\sqrt{1-\eta_0^2}\sin\psi}{\sqrt{1-\eta_0^2\sin^2\psi}}.$$

One verifies easily that $\cos 2\chi_0=\cos 2\chi$ and $\sin 2\chi_0=\sin 2\chi$. Thus in the limit $\delta=0$, the quantity χ_0 and thus the integral N_3 both reduce correctly.

7. Assembly of Equations for the Inversion

From (11), (67), (78), and (115) of the present paper and (5.30), (5.35), and (5.60) of V1961b, we have

$$t+\beta_1=(-2\alpha_1)^{-1/2}\ [b_1E+a(E-e\sin E)+A_1v+A_{11}\sin v+A_{12}\sin 2v]$$
$$+c^2\alpha_2^{-1}u^{1/2}\left[\left(\psi+\frac{\pi}{2}\right)B_1'+B_{11}\cos\psi+B_{12}\sin 2\psi+B_{13}\cos 3\psi+B_{14}\sin 4\psi\right] \qquad (116.1)$$

$$\beta_2'\alpha_2^{-1}=-(-2\alpha_1)^{-1/2}\left[A_2v+\sum_{k=1}^{4}A_{2k}\sin kv\right]+\alpha_2^{-1}u^{1/2}\left[\left(\psi+\frac{\pi}{2}\right)B_2+B_{21}\cos\psi\right.$$
$$\left.+B_{22}\sin 2\psi+B_{23}\cos 3\psi+B_{24}\sin 4\psi\right] \qquad (116.2)$$

where

$$B_{11}=2PQ-\frac{3}{8}C_1Q^3,\qquad B_{12}=-\left(\frac{Q^2}{4}+\frac{1}{8}C_2Q^4\right)$$
$$B_{13}=-C_1Q^3/24,\qquad B_{14}=C_2Q^4/64 \qquad (116.3)$$

$$B_{21}=C_2PQ-\frac{9}{16}C_1C_2Q^3+\frac{1}{2}C_1Q, \qquad B_{22}=-\frac{1}{32}[(4C_2+3C_1^2)Q^2+3C_2^2Q^4]$$

$$B_{23}=-\frac{1}{16}C_1C_2Q^3, \qquad B_{24}=\frac{3}{256}C_2^2Q^4 \tag{116.4}$$

and where B_2' and B_2 are given by (76) and (65).

Note that by V1962 the restrictions on p. 176 of V1961b, concerning orbital inclinations, have been removed.

Before proceeding further, it is convenient to eliminate the terms above involving $\pi/2$. If we lump the term $-c^2\alpha_2^{-1}u^{1/2}B_1'\pi/2$ into β_1, the physical meaning of β_1 will not be essentially changed. In the limit $J_2\to 0$, this change will vanish, since $c^2=O(J_2)$, and $-\beta_1$ will still reduce in the elliptic limit to τ, the time of passage through perigee. Also, for $J_2=0$ and $J_3=0$, $u=1$ and $B_2=1$, so that the term $-\alpha_2^{-1}u^{1/2}B_2\pi/2$ reduces to $-\frac{1}{2}\pi\alpha_2^{-1}$. Since $\beta_2'\to\omega+\pi/2$ in the elliptic limit, if we put

$$\beta_2=\beta_2'-u^{1/2}B_2\pi/2, \tag{117}$$

the new constant β_2 will reduce in the elliptic limit to ω, the argument of perigee.

With these changes, we then have

$$t+\beta_1=(-2\alpha_1)^{-1/2}[b_1E+a(E-e\sin E)+A_1v+A_{11}\sin v+A_{12}\sin 2v]$$

$$+c^2\alpha_2^{-1}u^{1/2}[B_1'\psi+B_{11}\cos\psi+B_{12}\sin 2\psi+B_{13}\cos 3\psi+B_{14}\sin 4\psi] \tag{118.1}$$

$$\beta_2\alpha_2^{-1}=-(-2\alpha)^{-1/2}\Big[A_2v+\sum_{k=1}^{4}A_{2k}\sin kv\Big]$$

$$+\alpha_2^{-1}u^{1/2}[B_2\psi+B_{21}\cos\psi+B_{22}\sin 2\psi+B_{23}\cos 3\psi+B_{24}\sin 4\psi]. \tag{118.2}$$

To carry out the inversion, we write

$$E=M_s+E_p, \qquad v=M_s+E_p, \qquad \psi=\psi_s+\psi_p, \tag{119}$$

where the subscript s means "secular" and the subscript p "periodic."

8. The Secular Solution

To obtain the secular terms M_s and ψ_s, we put $E=v=M_s$ in (118) and discard all the sines and cosines, thereby obtaining for M_s and ψ_s the pair of linear algebraic equations:

$$(-2\alpha_1)^{-1/2}(a+b_1+A_1)M_s+c^2\alpha_2^{-1}u^{1/2}B_1'\psi_s=t+\beta_1 \tag{120.1}$$

$$-(-2\alpha_1)^{1/2}A_2M_s+\alpha_2^{1/2}u^{1/2}B_2\psi_s=\beta_2\alpha_2^{-1}. \tag{120.2}$$

Their solution is

$$M_s=l_0+2\pi\nu_1t \qquad (121.1), \qquad \psi_s=l_0+g_0+2\pi\nu_2t, \tag{121.2}$$

where

$$2\pi\nu_1=(-2\alpha_1)^{1/2}(a+b_1+A_1+c^2A_2B_1'B_2^{-1})^{-1} \tag{122.1}$$

$$2\pi\nu_2 = \alpha_2 u^{-1/2} A_2 B_2^{-1}(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1})^{-1} \tag{122.2}$$

$$l_0 = 2\pi\nu_1(\beta_1 - c^2\beta_2\alpha_2^{-1}B_1' B_2^{-1}) \tag{123.1}$$

$$l_0 + g_0 = 2\pi\nu_2[\beta_1 + \beta_2\alpha_2^{-1}(a + b_1 + A_1)A_2^{-1}]. \tag{123.2}$$

9. The Fundamental Frequencies

By V1961a and pp. 188 and 189 of V1961b, the above ν's should satisfy

$$\nu_1 = \frac{\partial\alpha_1}{\partial j_1} \qquad (124.1) \qquad \nu_2 = \frac{\partial\alpha_1}{\partial j_2} \tag{124.2}$$

where j_1 and j_2 are the action variables corresponding to ρ and η, respectively. From V1961b, they should thus satisfy

$$\nu_1 = (j_{11}j_{22} - j_{12}j_{21})^{-1} j_{22}, \qquad \nu_2 = -(j_{11}j_{22} - j_{12}j_{21})^{-1} j_{12} \tag{125}$$

where

$$j_{11} = 2\pi(-2\alpha_1)^{-1/2}(a + b_1 + A_1), \qquad j_{12} = -2\pi\alpha_2(-2\alpha_1)^{-1/2} A_2 \tag{126}$$

$$j_{21} = 2c^2 N_1(\eta_0), \qquad j_{22} = 2\alpha_2 N_2(\eta_0). \tag{127}$$

Now when $\eta = \eta_0$, we have $\psi = \pi/2$, by (45.1), (47), and (48). Then, by (71), (78), and (127)

$$j_{21} = 2\pi c^2 \alpha_2^{-1} u^{1/2} B_1', \qquad j_{22} = 2\pi u^{1/2} B_2. \tag{128}$$

By means of (125) through (128), we readily verify that the fundamental frequencies $\partial\alpha_1/\partial j_1$ and $\partial\alpha_2/\partial j_2$ agree with the expressions given in (122).

10. Periodic Terms

In calculating the periodic terms through the second order, we first set

$$E_p = E_0 + E_1 + E_2, \qquad v_p = v_0 + v_1 + v_2, \qquad \psi_p = \psi_0 + \psi_1 + \psi_2, \tag{129}$$

where the subscript denotes the order of a term in J_2. We shall calculate the zeroth, first, and second order terms in succession.

Before we can do so, however, we need a table of the order of the various coefficients and constant quantities. Including the quantities necessary for the right ascension φ also, we find

Quantity	Order	Quantity	Order
A_2	J_2^0	B_3	J_2
B_1'	J_2^0	A_{11}	J_2^2
B_2	J_2^0	A_{12}	J_2^2
B_{12}	J_2^0	A_{23}	J_2^2
b_1	J_2	A_{24}	J_2^2
A_1	J_2	B_{21}	J_2^2
B_{11}	J_2	B_{23}'	J_2^2
A_{21}	J_2	B_{24}	J_2^2
A_{22}	J_2		
B_{13}	J_2		
B_{14}	J_2		
B_{22}	J_2		

10.1. Zeroth Order Periodic Terms

To find E_0, v_0, and ψ_0, place $E = M_s + E_0$, $v = M_s + v_0$, $\psi = \psi_s + \psi_0$ in (118), retain secular terms with full accuracy, and discard all periodic quantities of order higher than J_2^0. However, as cautioned in V1961b, keep all terms that are retained to as high an accuracy as possible; such a procedure simplifies the resulting formulas. Then

$$t + \beta_1 = (-2\alpha_1)^{-1/2}[(a + b_1)(M_s + E_0) - ae \sin(M_s + E_0) + A_1 M_s] + c^2 \alpha_2^{-1} u^{1/2} B_1' \psi_s \tag{129.1}$$

$$\beta_2 \alpha_2^{-1} = -(2\alpha_1)^{-1/2} A_2 (M_s + v_0) + \alpha_2^{-1} u^{1/2} B_2 (\psi_s + \psi_0). \tag{129.2}$$

Subtraction of (120.1) from (129.1) then gives

$$(a + b_1)E_0 - ae \sin(M_s + E_0) = 0. \tag{130}$$

On dividing (120) by $a + b_1$ and adding M_s to both sides of the result, we find a Kepler equation for $M_s + E_0$, viz,

$$M_s + E_0 - e' \sin(M_s + E_0) = M_s \tag{131.1}$$

$$e' \equiv \frac{ae}{a + b_1} < e < 1. \tag{131.2}$$

Solve this Kepler equation for $E = M_s + E_0$. Then find $v = M_s + v_0$ by the anomaly connections

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \qquad \sin v = \frac{(1 - e^2)^{1/2} \sin E}{1 - e \cos E}. \tag{132}$$

To find ψ_0, subtract (120.2) from (129.2) and solve for ψ_0. The result is

$$\psi_0 = \alpha_2 (-2\alpha_1)^{-1/2} u^{-1/2} A_2 B_2^{-1} v_0. \tag{133}$$

As in V1961b, this gives $\psi_0 = v_0 + O(J_2)$, but it is important to retain the coefficient with full accuracy, for later simplifications.

10.2. First Order Periodic Terms

We now place $E = M_s + E_0 + E_1$, $v = M_s + v_0 + v_1$, $\psi = \psi_s + \psi_0 + \psi_1$ in (118), discarding periodic terms of order J_2^2, and use our knowledge of M_s, ψ_s, E_0, v_0, and ψ_0 to find E_1, v_1, and ψ_1. From (118) we obtain

$$t + \beta_1 = (-2\alpha_1)^{-1/2} [(a + b_1)(M_s + E_0 + E_1) - ae \sin(M_s + E_0 + E_1) + A_1 (M_s + v_0)] + c^2 \alpha_2^{-1} u^{1/2} [B_1' (\psi_s + \psi_0) + B_{12} \sin(2\psi_s + 2\psi_0)] \tag{134.1}$$

$$\beta_2 \alpha_2^{-1} = -(-2\alpha_1)^{-1/2} [A_2 (M_s + v_0 + v_1) + A_{21} \sin(M_s + v_0) + A_{22} \sin(2M_s + 2v_0)] + \alpha_2^{-1} u^{1/2} [B_2 (\psi_s + \psi_0 + \psi_1) + B_{22} \sin(2\psi_s + 2\psi_0)]. \tag{134.2}$$

On subtracting (120.1) from (134.1), transposing some terms, and dividing through by $(-2\alpha_1)^{-1/2}$ $(a+b_1)$, we find:

$$E_0+E_1-e'\sin(M_s+E_0+E_1)=-(a+b_1)^{-1}[A_1v_0+c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}B_1'\psi_0$$
$$+c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}B_{12}\sin(2\psi_s+2\psi_0)], \qquad (135)$$

where e' is given by (131.2). By (133) the terms in v_0 and ψ_0 consolidate into $(A_1+c^2A_2B_1'B_2^{-1})v_0$ and (135) becomes

$$M_s+E_0+E_1-e'\sin(M_s+E_0+E_1)=M_s+M_1, \qquad (136.1)$$

where

$$M_1=-(a+b_1)^{-1}[(A_1+c^2A_2B_1'B_2^{-1})v_0+c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}B_{12}\sin(2\psi_s+2\psi_0)]. \qquad (136.2)$$

By p. 194 of V1961b, the solution for E_1 is then given by

$$E_1=\frac{M_1}{1-e'\cos(M_s+E_0)}-\frac{e'}{2}\frac{M_1^2\sin(M_s+E_0)}{[1-e'\cos(M_s+E_0)]^3}+O(J_2^3)\cdot \qquad (137)$$

To find v_1, insert the known values of $E=M_s+E_0+E_1$ and e into the anomaly connections (132) and solve for $v=M_s+v_0+v_1$.

To find ψ_1, subtract (120.2) from (134.2). The result is

$$\alpha_2^{-1}u^{1/2}[B_2(\psi_0+\psi_1)+B_{22}\sin(2\psi_s+2\psi_0)]=(-2\alpha_1)^{-1/2}[A_2(v_0+v_1)$$
$$+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]. \qquad (138)$$

By virtue of (133) the terms proportional to ψ_0 and v_0 cancel out. On solving for ψ_1, we then find

$$\psi_1=-B_{22}B_2^{-1}\sin(2\psi_s+2\psi_0)+\alpha_2(-2\alpha_1)^{-1/2}u^{-1/2}B_2^{-1}[A_2v_1$$
$$+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]. \qquad (139)$$

10.3. Second Order Periodic Terms

To find E_2, v_2, and ψ_2, insert $E=M_s+E_0+E_1+E_2$, $v=M_s+v_0+v_1+v_2$, and $\psi=\psi_s+\psi_0+\psi_1+\psi_2$ into (118), retaining all periodic terms of order J_2^2 and discarding all those of higher order. We obtain

$$t+\beta_1=(-2\alpha_1)^{-1/2}[(a+b_1)(M_s+E_0+E_1+E_2)-ae\sin(M_s+E_0+E_1+E_2)$$
$$+A_1(M_s+v_0+v_1)+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)]$$
$$+c^2\alpha_2^{-1}u^{1/2}[B_1'(\psi_s+\psi_0+\psi_1)+B_{11}\cos(\psi_s+\psi_0)+B_{12}\sin(2\psi_s+2\psi_0+2\psi_1)$$
$$+B_{13}\cos(3\psi_s+3\psi_0)+B_{14}\sin(4\psi_s+4\psi_0)]. \qquad (140.1)$$

$$\beta_2\alpha_2^{-1} = -(-2\alpha_1)^{-1/2}[A_2(M_s+v_0+v_1+v_2)+A_{21}\sin(M_s+v_0+v_1)+A_{22}\sin(2M_s+2v_0+2v_1)$$

$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]+\alpha_2^{-1}u^{1/2}[B_2(\psi_s+\psi_0+\psi_1+\psi_2)+B_{21}\cos(\psi_s+\psi_0)$$

$$+B_{22}\sin(2\psi_s+2\psi_0+2\psi_1)+B_{23}\cos(3\psi_s+3\psi_0)+B_{24}\sin(4\psi_s+4\psi_0)]. \qquad (140.2)$$

To solve for E_2, subtract (134.1) from (140.1). We find

$$(-2\alpha_1)^{-1/2}[(a+b_1)E_2-ae\sin(M_s+E_0+E_1+E_2)+ae\sin(M_s+E_0+E_1)+A_1v_1+A_{11}\sin(M_s+v_0)$$

$$+A_{12}\sin(2M_s+2v_0)]+c^2\alpha_2^{-1}u^{1/2}[B_1'\psi_1+B_{11}\cos(\psi_s+\psi_0)+B_{12}\sin(2\psi_s+2\psi_0+2\psi_1)$$

$$-B_{12}\sin(2\psi_s+2\psi_0)+B_{13}\cos(3\psi_s+3\psi_0)+B_{14}\sin(4\psi_s+4\psi_0)]. \qquad (141)$$

Now

$$\sin(M_s+E_0+E_1+E_2)-\sin(M_s+E_0+E_1)=E_2\cos(M_s+E_0+E_1)+O(J_2^3) \qquad (142.1)$$

and

$$c^2\sin(2\psi_s+2\psi_0+2\psi_1)-c^2\sin(2\psi_s+2\psi_0)=2c^2\psi_1\cos(2\psi_s+2\psi_0)+O(J_2^3), \qquad (142.2)$$

where E_2 and ψ_1 are in radians. On inserting (142) into (141) and solving for E_2, we find

$$E_2=\frac{M_2}{1-e'\cos(M_s+E_0+E_1)} \qquad (143.1)$$

$$M_2\equiv-(a+b_1)^{-1}[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)$$

$$+c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}\{B_1'\psi_1+B_{11}\cos(\psi_s+\psi_0)+2B_{12}\psi_1\cos(2\psi_s+2\psi_0)$$

$$+B_{13}\cos(3\psi_s+3\psi_0)+B_{14}\sin(4\psi_s+4\psi_0)\}]. \qquad (143.2)$$

Comparison with (8.45) of V1961b shows that a term in $\sin(2\psi_s+2\psi_0)$ is now missing. The reason is that, by using the full accuracy for B_{12} in the first order periodic terms, we have already included it in M_1.

To find v_2, insert $E=M_s+E_0+E_1+E_2$ in the anomaly connections (132) and solve for $v=M_s+v_0+v_1+v_2$.

To find ψ_2, subtract (134.2) from (140.2) to obtain

$$-(-2\alpha_1)^{-1/2}[A_2v_2+A_{21}\sin(M_s+v_0+v_1)-A_{21}\sin(M_s+v_0)$$

$$+A_{22}\sin(2M_s+2v_0+2v_1)-A_{22}\sin(2M_s+2v_0)$$

$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]$$

$$+\alpha_2^{-1}u^{1/2}[B_2\psi_2+B_{21}\cos(\psi_s+\psi_0)+B_{22}\sin(2\psi_s+2\psi_0+2\psi_1)$$

$$-B_{22}\sin(2\psi_s+2\psi_0)+B_{23}\cos(3\psi_s+3\psi_0)+B_{24}\cos(4\psi_s+4\psi_0)]=0. \qquad (144)$$

On inserting

$$A_{21}\sin(M_s+v_0+v_1)-A_{21}\sin(M_s+v_0)=A_{21}v_1\cos(M_s+v_0)+O(J_2^3)$$

$$A_{22}\sin(2M_s+2v_0+2v_1)-A_{22}\sin(2M_s+2v_0)=2A_{22}v_1\cos(2M_s+2v_0)+O(J_2^3)$$

$$B_{22}\sin(2\psi_s+2\psi_0+2\psi_1)-B_{22}\sin(2\psi_s+2\psi_0)=2B_{22}\psi_1\cos(2\psi_s+2\psi_0)+O(J_2^3)$$

into (144) and solving for ψ_2, we find

$$\psi_2=-B_2^{-1}[B_{21}\cos(\psi_s+\psi_0)+2B_{22}\psi_1\cos(2\psi_s+2\psi_0)+B_{23}\cos(3\psi_s+3\psi_0)+B_{24}\sin(4\psi_s+4\psi_0)]$$
$$+\alpha_2 u^{-1/2}(-2\alpha_1)^{-1/2}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)+2A_{22}v_1\cos(2M_s+2v_0)$$
$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]. \qquad (145)$$

This completes the solution for the E, v, and ψ and thus for the spheroidal coordinates ρ and η.

11. The Right Ascension

To find φ we have to insert (115) for N_3 and (5.60) of V1961b for R_3 into (11.3). Then

$$\varphi=\beta_3'-c^2\alpha_3(-2\alpha_1)^{-1/2}\left[A_3v+\sum_{j=1}^{4}A_{3j}\sin jv\right]$$
$$+\alpha_3\alpha_2^{-1}u^{1/2}\left\{(1-S)^{-1/2}\left[(h_1+h_2)\left\{\frac{\pi}{2}(1+2\zeta)^{-1/2}+\chi_0\right\}\right.\right.$$
$$\left.\left.+(h_1-h_2)\left\{-\frac{\pi}{2}(1+2\zeta)^{-1/2}+\chi_1\right\}\right]+\left(\frac{\pi}{2}+\psi\right)B_3-\frac{3}{4}C_1C_2Q\cos\psi+\frac{3}{32}C_2^2Q^2\sin 2\psi\right\}. \qquad (146)$$

Here A_3 is given by (5.61) through (5.65) in V1961b. Because of typographical errors there, I include here the correct expressions for A_{32} and A_{33}:

$$A_{32}=(1-e^2)^{1/2}p^{-3}\left[\frac{e^2}{4}+\frac{3}{4}\frac{b_1}{p}e^2-p^{-2}\left(\frac{e^4}{4}+\frac{3}{2}e^2\right)\left(\frac{b_2^2}{2}+e^2\right)\right]$$
$$A_{33}=(1-e^2)^{1/2}p^{-3}\left[\frac{e^3b_1}{12p}-\frac{e^3}{3p^2}\left(\frac{b_2^2}{2}+e^2\right)\right]. \qquad (146.1)$$

Here b_2^2 is simply the B in (34.1) of this paper.

The symbols h_1 and h_2 are abbreviations:

$$h_1\equiv\frac{1}{2}(1+C_1-C_2)^{-1/2} \qquad (147.1) \qquad h_2\equiv\frac{1}{2}(1-C_1-C_2)^{-1/2}. \qquad (147.2)$$

By (18), (21.2), (26), and (31), we find for α_3

$$\alpha_3=[\mu p_0(1-S)]^{1/2}\operatorname{sgn}\alpha_3\left[1-\frac{c^2S}{a_0p_0}-\frac{\left(\frac{2\delta}{p_0}\right)^2\left(1-\frac{c^2S}{a_0p_0}\right)S}{\left[1+\frac{c^2}{a_0p_0}(1-2S)\right]^2}\right]^{1/2}. \qquad (148)$$

The error here is of order J_2^4. With the same error, we can simplify this to

$$\alpha_3=[\mu p_0(1-S)]^{1/2}\operatorname{sgn}\alpha_3\left[1-S\left\{\frac{c^2}{a_0p_0}+\left(\frac{2\delta}{p_0}\right)^2\left(1-\frac{2c^2}{ap}+\frac{3c^2S}{ap}\right)\right\}\right]^{1/2} \qquad (149)$$

Alternatively, when α_2 is known, we may write

$$\alpha_3 = \alpha_2\left(1-\frac{S}{u}\right)^{1/2} \operatorname{sgn}\alpha_3. \tag{149.1}$$

In (146) the constant β_3' would reduce to $\Omega - \frac{1}{2}\pi \operatorname{sgn}\alpha_3$ in the Keplerian case, as one may see by solving the two body problem with lower limit $\eta_1(=-\eta_0)$ in the η-integrals. Also in (146), consider the constant term:

$$\tfrac{1}{2}\,\pi\alpha_3\alpha_2^{-1}u^{1/2}(1-S)^{-1/2}(1+2\zeta)^{-1/2}[h_1+h_2-(h_1-h_2)]+\tfrac{1}{2}\,\pi B_3.$$

For $J_2=0$ and $J_3=0$, we have $u=1$, $\zeta=0$, $h_1+h_2=1$, $h_1-h_2=0$, $B_3=0$, $S=\sin^2 I$, and $\alpha_3\alpha_2^{-1}=\cos I$. Thus it then becomes $\frac{1}{2}\,\pi\cos I|\sec I| = \frac{1}{2}\,\pi\operatorname{sgn}\alpha_3$. If we lump it into the constant β_3' and call the sum β_3, we then find that β_3 reduces, in the Keplerian case, to Ω, the right ascension of the node.

Our amended equation thus takes the form

$$\varphi = \beta_3 - c^2\alpha_3(-2\alpha_1)^{-1/2}\left[A_3 v+\sum_{j=1}^{4}A_{3j}\sin jv\right]+\alpha_3\alpha_2^{-1}u^{1/2}\left\{(1-S)^{-1/2}[(h_1+h_2)\chi_0\right.$$

$$\left.+(h_1-h_2)\chi_1]+B_3\psi-\frac{3}{4}\,C_1C_2Q\cos\psi+\frac{3}{32}\,C_2^2Q^2\sin 2\psi\right\}, \tag{150}$$

where χ_0 and χ_1 are given by (114) and E_2' and E_3' in (114) by (104), with $e_2=Q(1-P)^{-1}$ and $e_3=Q(1+P)^{-1}$.

It is not difficult to verify that (150) reduces to (8.50) of V1961b in the limit $\delta=0$. A much more delicate check is to show that (150) makes φ a constant in a polar orbit, with jumps of π as the satellite goes over a pole.

As an orbit approaches being a polar orbit, $\alpha_3\to 0$ and $S\to 1$, so that

$$\varphi\to\beta_3+\alpha_3\alpha_2^{-1}u^{1/2}[(h_1+h_2)\chi_0+(h_1-h_2)\chi_1](1-S)^{-1/2}. \tag{151}$$

Now by (148), since $(\mu p_0)^{1/2}=\alpha_2$, we have for $S\to 1$

$$\alpha_3\alpha^{-1}u^{1/2}(1-S)^{-1/2}=\operatorname{sgn}\,\alpha_3\left[1-\frac{c^2}{a_0p_0}-\frac{(2\delta/p_0)^2}{1-\frac{c^2}{a_0p_0}}\right]^{1/2}, \tag{152}$$

since by (31) $u\to 1$ as $S\to 1$. Also, for $S=1$, we have by (32) that (152) becomes

$$\alpha_3\alpha_2^{-1}u^{1/2}(1-S)^{-1/2}=\operatorname{sgn}\,\alpha_3[(1-C_2)^2-C_1^2]^{1/2}(1-C_2)^{-1/2}. \tag{153}$$

To simplify the term in (151) involving the χ's, note that by (102.2), (32.1), and (32.2)

$$2\zeta=(1-C_2)^{-1}C_1. \tag{154}$$

Then, by (147) and (154)

$$(1-2\zeta)^{-1/2}=2h_2(1-C_2)^{1/2} \qquad (155.1), \qquad (1+2\zeta)^{1/2}=2h_1(1-C_2)^{1/2}. \tag{155.2}$$

In (114) let us now abbreviate $E_2'(\psi+\pi/2)$ to E_2' and $E_3'(\psi-\pi/2)$ to E_3'. Equations (114) and (155) then yield

$$\chi_0=(1-C_2)^{1/2}(h_2E_2'+h_1E_3') \tag{156.1}$$

$$\chi_1=(1-C_2)^{1/2}(h_2E_2'-h_1E_3'), \tag{156.2}$$

from which it follows that

$$(h_1+h_2)\chi_0+(h_1-h_2)\chi_1=2(1-C_2)^{1/2}h_1h_2(E_2'+E_3') \tag{157}$$

$$=\tfrac{1}{2}(1-C_2)^{1/2}[(1-C_2)^2-C_1^2]^{-1/2}(E_2'+E_3'), \tag{158}$$

after insertion into (157) of the expressions (147) for h_1 and h_2. On inserting (153) and (158) into (151), we then find that for a polar orbit

$$\varphi\to\beta_3+\tfrac{1}{2}(E_2'+E_3')\ \text{sgn}\ \alpha_3. \tag{159}$$

To discuss this expression, note that $e_2=Q(1-P)^{-1}$ and $e_3=Q(1+P)^{-1}$ are both unity for a polar orbit, since by (32.3) $P=0$ when $S=1$ and since by (47) Q then equals $S=1$.

To see how $E_2'+E_3'$ behaves as $S\to 1$, we have to discuss the function $E'(f)$, where the argument f plays the role of true anomaly and the function E' the role of eccentric anomaly.

Now by their definitions E' and f agree in value at all multiples of π. Also

$$\frac{dE'}{df}=\frac{\sqrt{1-e^2}}{1+e\cos f}. \tag{160}$$

Thus

$$\frac{dE'}{df}=\left(\frac{1-e}{1+e}\right)^{1/2}\to 0\ \text{as}\ e\to 1, \qquad f=2\tau\pi \tag{161.1}$$

$$=\left(\frac{1+e}{1-e}\right)^{1/2}\to\infty\ \text{as}\ e\to 1, \qquad f=(2\tau+1)\pi \tag{161.2}$$

The plot of E' versus f for $e=1$ is thus as follows

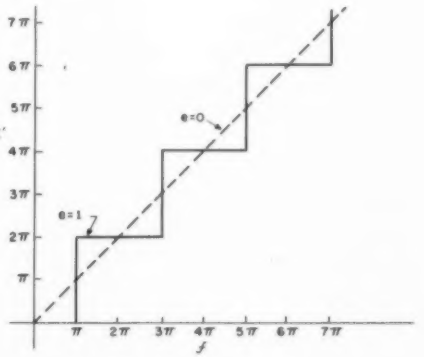

FIGURE 1. *Plot of eccentric anomaly versus true anomaly for e=1.*

For $e=1$, one sees that $E'(f)$ remains constant from $f=(2\tau+1)\pi$ to $(2\tau+3)\pi$, but then jumps by 2π.

Let us assume sgn $\alpha_3=1$ and consider what happens as ψ increases from $-\frac{\pi}{2}+\epsilon'$ to $\frac{\pi}{2}-\epsilon'$. In this range $\psi+\pi/2$ increases from ϵ' to $\pi-\epsilon'$ and $\psi-\pi/2$ from $-\pi+\epsilon'$ to $-\epsilon'$. In this range, therefore, both $E'(\psi+\pi/2)$ and $E'(\psi-\pi/2)$ remain constant, so that φ remains constant, by (159). As ψ increases beyond $\pi/2$, however, $\psi+\pi/2$ passes through an odd multiple of π and $\psi-\pi/2$ through an even multiple of π. Thus $E'(\psi+\pi/2)$ jumps by 2π and $E'(\psi-\pi/2)$ remains constant, so that, by (159), φ jumps by π. This is at the north pole. A similar jump takes place at the south pole. These results are of course trivial for a polar orbit, but they constitute an important test that the equation (150) for the right ascension had to pass.

12. The Final Algorithm

To summarize results, assume that μ, r_e, J_2, J_3, and the orbital elements a, e, S, l_0, g_0, and β_3 are given. (Here S is an element that would reduce to $\sin^2 I=\eta_0^2$ of V1961b, but with J_3 incorporated into the reference orbit it may go negative for orbits close to equatorial.)

To calculate the reference orbit, compute

$$c^2=r_e^2J_2\left(1-\frac{1}{4}J_3^2J_2^{-3}\right),\ \delta=-\frac{1}{2}r_eJ_2^{-1}J_3>0 \text{ for } J_3<0 \qquad p=a(1-e^2),$$

A by (40.1), and B by (40.2). Then

$a_0=a-\frac{1}{2}A$, $p_0=a_0^{-1}(B+ap-2Aa-c^2)$, $\alpha_2=(\mu p_0)^{1/2}$, u by (31), C_2 by (32.2), C_1 by (31.1), P by (32.3), $b_1=-\frac{1}{2}A$, $b_2=B^{1/2}$, $\alpha_1=-\frac{1}{2}\mu a_0^{-1}$. Then compute

$A_1, A_2, A_3, A_{11}, A_{12}, A_{21}, A_{22}, A_{23}, A_{24}, A_{31}, A_{32}, A_{33}, A_{34}$ by p. 198 of V1961b.

Now compute $Q=(P^2+S)^{1/2}$, B_2 by (65), B_1' by (76), B_3 by (95), ζ by (102.2),

$$e_2=Q(1-P)^{-1} \qquad e_3=Q(1+P)^{-1}$$

$2\pi\nu_1$ by (122.1), $2\pi\nu_2$ by (122.2), and l_0 and g_0 by (123). Then

$$M_s=l_0+2\pi\nu_1t, \qquad \psi_s=l_0+g_0+2\pi\nu_2t$$

$$e'=\frac{ae}{a+b_1}<e<1.$$

Find M_s+E_0 by solving

$$M_s+E_0-e'\sin(M_s+E_0)=M_s.$$

Solve (132) for $v=M_s+v_0$ and (133) for ψ_0. Then find M_1 by (136.2) and E_1 by (137) and solve (132) for $v=M_s+v_0+v_1$. Find ψ_1 by (139) and then M_2 by (143.2) and E_2 by (143.1).

Solve (132) for $v=M_s+v_0+v_1+v_2$ and find $\tilde{\psi}_2$ by (145). Then

$$E=M_s+E_0+E_1+E_2,\ v=M_s+v_0+v_1+v_2,\ \psi=\psi_s+\psi_0+\psi_1+\psi_2.$$

The spheroidal coordinates ρ and η are then given by

$$\rho = a(1 - e \cos E) = (1 + e \cos v)^{-1} p$$

$$\eta = P + Q \sin \psi.$$

To find the right ascension φ, note first the correct formulas for A_{32} and A_{33} in (146.1) of this paper or on p. 198 of V1961b. Then compute h_1 and h_2 by (147) and α_3 by (148). Define $E_2'(y)$ by

$$\cos E_2'(y) = \frac{e_2 + \cos y}{1 + e_2 \cos y} \qquad \sin E_2'(y) = \frac{(1 - e_2^2)^{1/2} \sin y}{1 + e_2 \cos y}$$

and the requirement that E_2' and y shall be equal whenever y is a multiple of π. Define $E_3'(y)$ in the same way, with e_3 replacing e_2.

Then calculate χ_0 and χ_1 by (114) and φ by (150). Having ρ, η, and φ, we may now find the rectangular and spheroidal coordinates by means of (1.1) and (1.2).

13. Further Remarks

The present inclusion of J_3 into the separable orbit has apparently led to new secular terms of order J_2^2. These terms, $-\frac{1}{2} C_1 P + \frac{3}{16} C_1^2 Q^2$ in B_2 and $-\frac{3}{8} C_1^2$ in B_3, are proportional to $J_3^2 J_2^{-2} = O(J_2^2)$ and lead to terms in M_s, ψ_s, and $\dot{\bar{\varphi}}$. Since no previous treatment of third harmonic perturbations has led to such secular terms, the result requires checking.

To do so, note that by (2) and (3) the present c^2 is given by

$$c^2 = c_0^2 - \delta^2$$

where $c_0^2 = r_1^2 J_2$. It turns out that the δ^2 terms implicit in the new c^2 are just sufficient to cancel the new terms in δ^2.

Since $c^4 = c_0^4 - 2c_0^2\delta^2 + \delta^4$, terms of order c^4 cannot cancel terms in δ^2. In expanding the fundamental frequencies it is thus sufficient to carry terms only through c^2 and δ^2, in checking this point. A rather long calculation then shows that

$$\dot{M}_s = 2\pi\nu_1 = n_0 + O(c^4)$$

$$\dot{\psi}_s = 2\pi\nu_2 = n_0 \left[1 + \frac{3}{4}\frac{c_0^2}{p^2}(4 - 5S)\right] + O(c^4)$$

$$\dot{\bar{\varphi}} = 2\pi\nu_3 = n_0 \operatorname{sgn} \alpha_3 \left[1 + \frac{3}{4}\frac{c_0^2}{p^2}(4 - 5S) - \frac{3}{2}\frac{c_0^2}{p^2}(1 - S)^{1/2}\right] + O(c^4).$$

Here $n_0 = \mu^{1/2} a_0^{-3/2}$, the mean motion corresponding to $a_0 = -\mu(2\alpha_1)^{-1}$, where α_1 is the total energy. This means that, if we solve the problem for given initial conditions without using the third harmonic and then re-solve it with inclusion of the third harmonic, there will arise no secular terms of order $J_3^2 J_2^{-2}$, provided that we use the same initial total energy. In turn, this means that if we use the same initial coordinates, we must change the initial velocities slightly or that if we use the same initial velocities we must change the initial coordinates slightly.

It does not appear worth while to do the extra calculation that would give us final formulas manifestly free of secular terms of order $J_3^2 J_2^{-2}$. If we use the present results, we shall obtain secular terms correct through order J_2^3; this limitation comes from the error of order J_2^4 in (31.2). To obtain this secular accuracy, we find from (122) and (150) that we should keep terms through order

J_2^3 in B_2 and B_3 and through order J_2^2 in B_1'. By improving the solution of the cubic (27), we could calculate secular terms to any order of accuracy. Periodic terms are correct through order J_2^2; they also may be carried to a higher order if the need ever arises.

To obtain the orbital elements one may proceed as follows. From initial values determine α_1, α_2, and α_3 by the appropriate modifications of eqs (3.8), (3.9), and (3.10) of V1961. They are

$$\alpha_1 = \tfrac{1}{2}\, u_i^2 - \mu(\rho_i + \eta_i\delta)\,(\rho_i^2 + c^2\eta_i^2)^{-1}$$

$$\alpha_3 = r_i^2 \cos^2 \theta_i \dot{\varphi}_i = x_i\dot{y}_i - y_i\dot{x}_i$$

$$\alpha_2^2 = (1-\eta_i^2)^{-1}[(\rho_i^2 + c^2\eta_i^2)^2\dot{\eta}_i^2 + \alpha_3^2 - (1-\eta_i^2)\,(2\alpha_1 c^2\eta_i^2 + 2\mu\eta_i\delta)].$$

Here the equation for α_2 follows from (10.2), (13.2), (50), and (59.1) of V1959. The subscript i means initial value and u_i is the initial speed.

From the α's one can determine a_0, e_0, p_0, and i_0 from eqs (21) and (28) of the present paper and then determine a, e, and $\eta_0 = \sin I$ by means of the equations on pp. (172) and (173) of V1961b. Knowing a, e, and I to this approximation, one can then determine the values of β_1, β_2, and β_3 or of l_0, g_0, and β_3 by fitting the values of ρ, η, and φ at various times, still using the equations in V1961b.

To improve these elements, now calculate the orbit, with elements a, e, $S = \sin^2 I$, l_0, g_0, and β_3 by means of the present paper. Observe the residuals and use them, with the aid of least squares, to obtain a differential correction to improve the orbital elements. Here a word of caution is necessary. If the orbit is close to equatorial, S may go negative. This is all right, since it is only $Q^2 = P^2 + S$ that cannot become negative, with the present method.

Then repeat the process, until the elements converge to fixed values. One or two iterations should suffice, according to Bonavito (1964).

The present method of accounting for J_3 ought to be an improvement over the perturbation theory of V1963, for two reasons. There appear no small denominators, such as those containing e or $\sin I$ in the perturbation theory. All periodic terms are correct through order J_2^2.

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(Paper 70B1–166)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

On *EPr* and Normal *EPr* Matrices[1]

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(September 30, 1965)

(1) *EPr* matrices A (that is, matrices A for which A and A^* have the same null space) are investigated. It is shown that if A is a complex EPr_1 matrix and B a complex EPr_2 matrix, and $AB=BA$, then AB is EPr. Other theorems about products of *EPr* matrices are established.

(2) Let A be a normal *EPr* matrix over an arbitrary field. A necessary and sufficient condition, involving the solvability (for X) of a matrix equation $XBX^*+AX+X^*A^*+C=0$, is found for the existence of a matrix N such that (i) $NN^*=I$ and (ii) $A^*=NA=AN$. Explicit solutions are given for two important classes of normal *EPr* matrices, namely (1) those satisfying the condition rank $A=$ rank AA^*, and (2) those of rank $n/2$, satisfying $AA^*=0$, over a field of characteristic $\neq 2$. An example is given to show that no such N need exist for characteristic $=2$.

(3) *EP* linear transformations on a finite-dimensional vector space are introduced, and the relation between them and *EPr* matrices is studied. It is shown that a linear transformation T of a complex vector space is *EP* if and only if rank $T=$ rank T^2.

Key Words: Matrix, normal, linear algebra.

1. Introduction

The concept of a normal matrix with entries from the complex field was introduced in 1918 by O. Toeplitz [19][3] who gave a necessary and sufficient condition that a complex matrix be normal. Since that time normal matrices and the generalization to linear transformations on a finite dimensional or infinite dimensional vector space [3, 7, 9, 18, 22, 23], to functional analysis, especially Hilbert space [5, 12, 13, 14], and to combinatorial analysis and the study of finite projective planes [1, 2, 4] have received a great deal of attention. Also, special types of normal matrices and linear transformations have been studied. But, until the appearance of [10] in 1959, no study had been made of normal matrices without restrictions on the underlying field.

First, results about *EPr* complex matrices, a concept introduced by H. Schwerdtfeger in [15] as a generalization of normality, were obtained and then in [10, 11] the notion of *EPr* was extended to matrices over arbitrary fields and applied to obtain results about normal matrices. One interesting feature of the study was the discovery that over an arbitrary field the concepts of normal and *EPr* are independent and that many of the well-known properties of complex normal matrices which do not carry over to an arbitrary field appear to generalize most naturally to matrices that are both *EPr* and normal. A matrix A of rank r is called *EPr* if A and A^* have the same null space.

The first section of this paper is concerned solely with developing the properties of *EPr* matrices. Real and complex *EPr* matrices are studied for their own inherent interest and a number of new results have been obtained. It is also shown how these results depend on the underlying field. Finally the structure of *EPr* matrices over an arbitrary field is developed, primarily for its use as a tool in section 3, which is devoted to normal matrices.

In section 3, a resumé of the known results concerning normal matrices over an arbitrary field is given. Then some questions raised in [10, 11] are considered and the concept of zero-type *EPr* matrix is introduced. For this new class of matrices a satisfactory solution is obtained to a problem dealt with in [10, 11]. Most of the results are partial results and interesting questions still remain to be answered.

[1] This research was supported by the National Science Foundation, under research grant NSF-GP 157.

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[3] Figures in brackets indicate the literature references at the end of this paper.

The material in section 4 is an outgrowth of an attempt to bring to bear on the study at hand the tools and methods of linear transformation theory. The concept of an *EP* linear transformation is introduced and related to the notion of *EPr* matrix. Various results are given, indicating how the study of these special linear transformations will yield results about *EPr* matrices. The concept of a θ-normal linear transformation, generalizing the concept of a normal linear transformation, is introduced and exploited to obtain anew the results of [10, 11].

We have tried to make this paper complete, readable and self-contained. It is hoped that because of our exposition here, some of the questions which we have left open (see sec. 5) will be resolved in the near future.

2. Structure Theory of *EPr* Matrices

1. In this section we develop the structure theory of *EPr* matrices over an arbitrary field F, essentially as given in [10]. However, in order to indicate clearly the difficulties encountered in extending results from the complex field to an arbitrary field because of the lack of a spectral theorem, to place in a proper setting those results which have not been extended, and to provide new results about *EPr* matrices with complex entries which give rise to additional questions over an arbitrary field, we begin with a comprehensive treatment of the complex case, basing the discussion on [10].

2. We begin by defining the notion of *EPr* matrix over the complex field.

Except for denoting the $k \times k$ identity matrix by I_k, subscripts on matrices will be used only to designate a row of a matrix; that is, A_i is the ith row of the matrix A. A^i will denote the ith column of A; when a superscript denotes a power of a matrix, the meaning will always be clear from the context.

DEFINITION: An $n \times n$ matrix A with entries from the complex field $\mathscr{C}$ is called *EPr* if it satisfies the following conditions.

(1) A has rank r.

(2) $\sum_{i=1}^{n} \alpha_i A_i = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0$ $(\alpha_i \epsilon \mathscr{C}, i = 1, \ldots, n)$.

As indicated in the introduction, this paper is concerned with matrices that are both *EPr* and normal and whose elements are taken from an arbitrary field. Hence we note immediately that these concepts are not independent over $\mathscr{C}$. After proving theorem 1.1 which gives eight necessary and sufficient conditions that a matrix be *EPr*, we shall be able to give a simple proof that any normal complex matrix of rank r is *EPr*. The converse statement is false, though, since all nonsingular $n \times n$ matrices are necessarily *EPn*. For, assuming the $n \times n$ matrix A is nonsingular,

$$\sum_{i=1}^{n} \alpha_i A_i = 0$$

if and only if

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

if and only if

$$\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0.$$

A portion of this next result is known [15], but is included for completeness. The proof presented is a slight modification and expansion of that found in [10].

THEOREM 1.1: *The following statements are equivalent.*

(1) A *is an* $n \times n$ EP_r *matrix,*

(2) A *is unitarily similar to the direct sum of a nonsingular* $r \times r$ *matrix* D *and a zero matrix. That is,*

$$UAU^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where U *satisfies* $UU^* = I$ *and* D *is* $r \times r$ *and nonsingular.*

(3) A *is conjunctive to the direct sum of a nonsingular* $r \times r$ *matrix* D *and a zero matrix. That is*

$$QAQ^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where Q *is nonsingular,* D *is* $r \times r$ *and nonsingular.*

(4) A *is the matrix of a linear transformation* T *acting on* $\mathscr{C}_n$, *complex* n-*dimensional Euclidean space, and there are mutually orthogonal subspaces* V_1 *and* V_2 *of* $\mathscr{C}_n$ *such that* V_1 *has dimension* r,

$$T(V_1) = V_1,\ T(V_2) = 0$$

and

$$\mathscr{C}_n = V_1 \oplus V_2.$$

(5) A *has rank* r *and there is an* $n \times n$ *matrix* N *such that* $A^* = NA$.

(6) A *has rank* r *and there is a nonsingular* $n \times n$ *matrix* N *such that* $A^* = NA$.

(7) A *can be represented as*

$$A = P\left[\begin{array}{c|c} D & DX^* \\ \hline XD & XDX^* \end{array}\right]P^* = P\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I \end{array}\right]P^*$$

where P *is a permutation matrix and* D *is an* $r \times r$ *nonsingular matrix.*

(8) $A\xi = 0$ *if and only if* $A^*\xi = 0$ *where* $\xi \in \mathscr{C}_n$.

(9) $R(A) = R(A^*)$; *that is,* A *and* A^* *have the same range spaces.*

PROOF: The implications are proved in the following order:

(8) $\Rightarrow$ (1) $\Rightarrow$ (7) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (8);
(1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1);
(5) $\Rightarrow$ (9) $\Rightarrow$ (5).

(8) $\Rightarrow$ (1). Suppose $\sum_{i=1}^{n} \alpha_i A_i = 0$ $(\alpha_i \in \mathscr{C},\ i = 1, 2, \ldots, n)$.

Let

$$\xi = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \in \mathscr{C}_n.$$

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Then

$$\xi^t A = [\alpha_1 \alpha_2 \ . \ . \ . \ \alpha_n] A = \sum_{i=1}^{n} \alpha_i A_i = 0.$$

Hence, taking the conjugate transpose of this equation, it follows that

$$A^* \bar{\xi} = 0.$$

Applying (8) we have

$$A \bar{\xi} = 0.$$

Thus

$$\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0.$$

The implication in the other direction is proved similarly.

(1) $\Rightarrow$ (7). Let A be EPr and let the rows $A_{i_1}, A_{i_2}, \ . \ . \ ., A_{i_r}$ be linearly independent. If

$$\sum_{k=1}^{r} \beta_{i_k} A^{i_k} = 0,$$

then

$$\sum_{k=1}^{r} \bar{\beta}_{i_k} A_{i_k} = 0$$

and hence $\bar{\beta}_{i_1} = \bar{\beta}_{i_2} \ . \ . \ . = \bar{\beta}_{i_r} = 0$. Thus $\beta_{i_1} = \beta_{i_2} = \ . \ . \ . = \beta_{i_r} = 0$ and so the columns $A^{i_1}, A^{i_2}, \ . \ . \ ., A^{i_r}$ are linearly independent. Since the rank of A is r, the submatrix D formed by the elements in the intersection of rows $A_{i_1}, A_{i_2}, \ . \ . \ ., A_{i_r}$ and the columns $A^{i_1}, A^{i_2}, \ . \ . \ ., A^{i_r}$ is an $r \times r$ nonsingular matrix [15, p. 52]. This "crossing theorem" can be quickly proved as follows. There is a permutation matrix $\tilde{P}$ such that A is brought into the form

$$B = \tilde{P} A \tilde{P}^* = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right]$$

by premultiplying A by $\tilde{P}$ and postmultiplying by $\tilde{P}^*$. Since the first block row of B is of the same rank r as is A and thus B, there is an $(n-r) \times r$ matrix H such that

$$[FG] = H[D, E],$$

and hence, by (1), such that

$$\begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} D \\ F \end{bmatrix} H^*.$$

If D had rank $< r$, then $D\xi = 0$ would hold for some $\xi \epsilon \mathscr{C}_r$, and so

$$\begin{bmatrix} D \\ F \end{bmatrix} \xi = \begin{bmatrix} D\xi \\ HD\xi \end{bmatrix} = 0,$$

contradicting the assumption that the first block column of B has rank r.

With H as above, let

$$\tilde{Q} = \left[\begin{array}{c|c} I_r & 0 \\ \hline -H & I_{n-r} \end{array}\right].$$

It is readily verified that

$$\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] = \tilde{Q}B\tilde{Q}^* = \tilde{Q}\tilde{P}A\tilde{P}^*\tilde{Q}^*$$

so [4]

$$A = \tilde{P}^{-1}\tilde{Q}^{-1}\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\tilde{Q}^{-*}\tilde{P}^{-*}$$

$$= \tilde{P}^{-1}\left[\begin{array}{c|c} I_r & 0 \\ \hline -\tilde{X} & I \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & -\tilde{X}^* \\ \hline 0 & I \end{array}\right]\tilde{P}^{-*}.$$

Set $P = \tilde{P}^{-1}$, $X = -\tilde{X}$, $Q = \tilde{Q}^{-1}$.

(7) $\Rightarrow$ (3). Set

$$Q = \left[\begin{array}{c|c} I_r & 0 \\ \hline X & I \end{array}\right].$$

(3) $\Rightarrow$ (6). If

$$QAQ^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where Q is nonsingular, D is $r \times r$ nonsingular, then

$$A^* = Q^{-1}\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0 \end{array}\right]Q^{-*} = Q^{-1}\left[\begin{array}{c|c} D^*\ D^{-1} & 0 \\ \hline 0 & I \end{array}\right]QQ^{-1}\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]Q^{-*} = NA$$

where

$$N = Q^{-1}\left[\begin{array}{c|c} D^*\ D^{-1} & 0 \\ \hline 0 & I \end{array}\right]Q$$

and N is nonsingular.

[4] We use the notation $\tilde{P}^{-*} = (\tilde{P}^{-1})^* = (\tilde{P}^*)^{-1}$.

(6) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (8). Let $A^* = NA$. Let $\eta(A)$ denote the null space of A and let $\xi \epsilon \eta(A)$, i.e., $A\xi = 0$. Then $A^*\xi = 0$. Thus $\eta(A) \subseteq \eta(A^*)$. But, since rank A = rank A^*, it follows that dim $\eta(A) = n -$ rank A $= n$-rank $A^* =$ dim $\eta(A^*)$ and so $\eta(A) = \eta(A^*)$. Thus if $A^*\xi = 0$, then $A\xi = 0$.

(1) $\Rightarrow$ (2). By the well-known result of Schur [17], there is a unitary matrix V such that

$$B = VAV^* = \begin{bmatrix} & * \\ & \ddots \\ 0 & \end{bmatrix},$$

i.e., B is lower triangular. Moreover, the diagonal elements may be arranged in any order. We assume that the nonzero terms precede the zero terms on the diagonal of B. If $r = n$, there is nothing more to prove. Hence we assume $r < n$. Then 0 is a characteristic root of A and appears on the diagonal of B. Thus the last row of VAV^* is a row of zeros. Hence

$$0 = 0 \cdot B_1 + \ldots + 0 \cdot B_{n-1} + 1 \cdot B_n$$

Since B is EPr we have $0 = 0 \cdot B^1 + \ldots + 0 \cdot B^{n-1} + 1 \cdot B^n$. Repeating this argument shows that $B_i = 0$ if and only if $B^i = 0$ and so

$$VAV^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is $r \times r$ and nonsingular.

(2) $\Rightarrow$ (4) Let

$$UAU^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $UU^* = I$ and D is $r \times r$ and nonsingular. Suppose

$$U = [u_{ij}] (1 \leqq i, j \leqq n)$$

and

$$D = [d_{ij}] (1 \leqq i, j \leqq r).$$

Let $e_1, e_2, \ldots, e_n$ denote the "natural basis" of $\mathscr{C}_n$, that is

$$e_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \\ \vdots \\ \delta_{in} \end{bmatrix} (i = 1, \ldots, n)$$

where δ_{ij} is the Kronecker delta. Define the following linear transformations:

$$Re_i = \sum_{\lambda=1}^{r} d_{\lambda i} e_\lambda \qquad (i = 1, \ldots, r)$$

$$Re_i = 0 \qquad (i = r+1, \ldots, n)$$

$$Se_i = \sum_{\lambda=1}^{n} u_{\lambda i} e_\lambda \qquad (i = 1, \ldots, n).$$

$$T = SRS^{-1}$$

Then S is unitary, that is, preserves the Euclidean inner product and the matrix of T is

$$U^{-1} \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] U = A$$

relative to $e_1, \ldots, e_n$. Set

$$\overline{V}_1 = < e_1, \ldots, e_r >,$$

$$\overline{V}_2 = < e_{r+1}, \ldots, e_n >,$$

the spaces spanned by $e_1, \ldots, e_r$ and $e_{r+1}, \ldots, e_n$ respectively. Then set $V_1 = S\overline{V}_1$ and $V_2 = S\overline{V}_2$. Hence

$$TV_1 = T(S\overline{V}_1) = SR\overline{V}_1 = S\overline{V}_1 = V_1$$

and

$$TV_2 = T(S\overline{V}_2) = SR\overline{V}_2 = S(0) = 0$$

Finally, $0 = (\bar{v}_1, \bar{v}_2) = (S\bar{v}_1, S\bar{v}_2)$, $\bar{v}_i \epsilon \overline{V}_i (i = 1, 2)$ so that V_1 and V_2 are orthogonal subspaces.

(4)$\Rightarrow$ (1). We prove this by showing that (4) implies (8). Suppose

$$A\xi = 0, \qquad \xi \epsilon \mathscr{C}_n, \qquad \xi = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Let $x_1, x_2, \ldots, x_n$ be the basis for $\mathscr{C}_n$ relative to which the matrix of T is A. Let V_1 have basis $v_1, \ldots, v_r$ and V_2 have basis $v_{r+1}, \ldots, v_n$. $A\xi = 0$ if and only if $\sum_{i=1}^{n} \alpha_i x_i \epsilon V_2$. Then, for $1 \leqq j \leqq r$,

$$\left(v_j, T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)\right) = \left(Tv_j, \sum_{i=1}^{n} \alpha_i x_i\right) = 0$$

since $Tv_j \epsilon V_1$ and $\sum_{i=1}^{n} \alpha_i x_i \epsilon V_2$, and for $r+1 \leqq j \leqq n$

$$\left(v_j, T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)\right) = \left(Tv_j, \sum_{i=1}^{n} \alpha_i x_i\right) = \left(0, \sum_{i=1}^{n} \alpha_i x_i\right) = 0.$$

Thus $T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)$ is in the orthogonal complement of $\mathscr{C}_n$, which is (0).

Hence $T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)=0$ so that $A^*\xi=0$.

(5)$\Rightarrow$ (9). $A^*x=NAx$ so $R(A)\subseteq R(A^*)$; since rank A = rank A^*, the equality follows.

(9)$\Rightarrow$ (5). This follows from [16], p. 92.

COROLLARY 1: *If the complex matrix* A *is* EPr, *then* A^n *is* EPr $(n=1, 2, \ldots)$.

PROOF: Apply (2). If

$$UAU^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

then

$$UA^nU^*=\left[\begin{array}{c|c} D^n & 0 \\ \hline 0 & 0 \end{array}\right].$$

COROLLARY 2: *Every matrix is a product of* EP *matrices.*[5]

PROOF: It is well known that an $n\times n$ matrix of rank r may be written as

$$P\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]Q$$

where P and Q are nonsingular.

COROLLARY 3: *A complex normal matrix* A *is* EP.

PROOF: Since A is normal there is a unitary matrix U such that UAU^* is diagonal [15], and hence (2) of the theorem may be applied.

We shall return to such questions in section 3 and see that the situation is quite different over arbitrary fields.

An example is given in subsection 4 following, to show that corollary 3 need not be valid over an arbitrary field.

3. If A is an EPr_1 matrix and B is an EPr_2 matrix, then AB may not be an EP matrix. For example, over $\mathscr{C}$,

$$A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

are normal, hence EP_1. But the product

$$C=AB=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

is not EP_1 since $C\begin{pmatrix}1\\0\end{pmatrix}=0$, but $C^*\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$.

We shall prove that commutativity of A and B alters the above result.

For $V\subseteq\mathscr{C}_n$, let $V^\perp$ denote the orthogonal complement of V under the usual inner product. Also, let $\eta(A)$, $\eta(B)$, and $\eta(AB)$ be the null spaces of A, B, and AB respectively.

[5] A matrix will be called *EP* if it is *EPr* for some r.

LEMMA 1: *Let* A *and* B *be complex* n × n *matrices satisfying* AB = BA. *Then*

$$A^*B^*\mathscr{C}_n \subseteq [\eta(A)+\eta(B)]^\perp$$

PROOF: Let $v\epsilon\mathscr{C}_n$, $x\epsilon\eta(A)$, $y\epsilon\eta(B)$.

$$(A^*B^*v,\ x+y)=(v,\ BAx+ABy)=(v,\ 0)=0$$

so that $A^*B^*v\epsilon[\eta(A)+\eta(B)]^\perp$

LEMMA 2: *With* A *and* B *as in lemma* 1, $A^*B^*\mathscr{C}_n \subseteq \eta(AB)^\perp$
PROOF: Let $v\epsilon\mathscr{C}_n$, $z\epsilon\eta(AB)$.

$$(A^*B^*v,\ z)=(v,\ BAz)=(v,\ 0)=0$$

so $A^*B^*v\epsilon\ \eta(AB)^\perp$.

LEMMA 3: *Again, take* A *and* B *as in lemma* 1. *Then*

$$B^*(\eta(A)^\perp) \subseteq \eta(A)^\perp$$

PROOF: Let $x\epsilon\eta(A)^\perp$, $y\epsilon\eta(A)$.

$$ABy=B(Ay)=0$$

so $By\epsilon\eta(A)$. Hence

$$(B^*x,\ y)=(x,\ By)=0$$

and $B^*x\epsilon\eta(A)^\perp$

THEOREM 1.2: *Let* A *be a complex* EPr_1 *matrix and* B *a complex* EPr_2 *matrix satisfying* AB = BA. *Then*

$$\eta(A)+\eta(B)=\eta(AB).$$

PROOF: Clearly $\eta(A)+\eta(B) \subseteq \eta(AB)$. The reverse inclusion is obtained by showing that

$$[\eta(A)+\eta(B)]^\perp \subseteq \eta(AB)^\perp.$$

First note that

$$\eta(A) \subseteq \eta(A)+\eta(B)$$

so

$$[\eta(A)+\eta(B)]^\perp \subseteq \eta(A)^\perp.$$

Similarly, $[\eta(A)+\eta(B)]^\perp \subseteq \eta(B)^\perp$ and thus

$$[\eta(A)+\eta(B)]^\perp \subseteq \eta(A)^\perp \cap \eta(B)^\perp.$$

Denote $\eta(A)^\perp \cap \eta(B)^\perp$ by W.

We wish to show that B is one-one on $\eta(B)^\perp$, that is, if $x_1, x_2 \in \eta(B)^\perp$ and $Bx_1 = Bx_2$ then $x_1 = x_2$. If $B(x_1 - x_2) = 0$ then $x_1 - x_2 \in \eta(B) \cap \eta(B)^\perp$. However $\eta(B) \cap \eta(B)^\perp = (0)$ because, with the usual inner product, $\mathscr{C}_n$ has no isotropic vectors $\neq 0$. Similarly, it follows that A is one-one on $\eta(A)^\perp$.

Now we may prove that A^*B^* is one-one on W. If $x_1, x_2 \in W$ and $A^*B^*(x_1 - x_2) = 0$, then $A^*(B^*(x_1 - x_2)) = 0$ and since A is EPr it follows that $A(B^*(x_1 - x_2)) = 0$. Thus $B^*(x_1 - x_2) \in \eta(A)$. But by lemma 3, $B^*(x_1 - x_2) \in \eta(A)^\perp$ since $x_1 - x_2 \in \eta(A)^\perp$. Therefore $B^*(x_1 - x_2) = 0$ so that $B(x_1 - x_2) = 0$ and since $x_1 - x_2 \in \eta(B)^\perp$, it follows that $x_1 - x_2 = 0$.

Also $A^*B^*[\eta(A) + \eta(B)]^\perp \subseteq A^*B^*\mathscr{C}_n \subseteq [\eta(A) + \eta(B)]^\perp$ by lemma 1, and because A^*B^* is one-one on W we have

$$A^*B^*[\eta(A) + \eta(B)]^\perp = [\eta(A) + \eta(B)]^\perp.$$

Finally,

$$[\eta(A) + \eta(B)]^\perp = A^*B^*[\eta(A) + \eta(B)]^\perp \subseteq A^*B^*\mathscr{C}_n \subseteq \eta(AB)^\perp$$

by lemma 2, completing the proof.

It is not needed in the above proof, but we can also show that $[\eta(A) + \eta(B)]^\perp = W$. For

$$\begin{aligned} A^*B^*W &\subseteq [\eta(A) + \eta(B)]^\perp \qquad \text{(by lemma 1)} \\ &\subseteq W \end{aligned}$$

and since A^*B^* is one-one on W it follows that $A^*B^*W = W$.

THEOREM 1.3: *If* A *is a complex* EPr_1 *matrix and* B *is a complex* EPr_2 *matrix such that* AB = BA, *then* AB *is an* EPr *matrix.*

PROOF: Suppose $ABz = 0$. Then $z \in \eta(AB) = \eta(A) + \eta(B)$. Hence there exist $x \in \eta(A)$, $y \in \eta(B)$ such that $z = x + y$. Furthermore since $\eta(A) = \eta(A^*)$ and $\eta(B) = \eta(B^*)$ we have

$$B^*A^*z = B^*A^*x + A^*B^*y = 0 + 0 = 0$$

which completes the proof.

A theorem of N. Wiegmann [20, 9] asserts that the normality of the complex matrices A, B, and AB implies the normality of BA. The corresponding result about EPr matrices is false. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are EP_2 and

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is EP_1. However,

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has rank 1 and is not EP_1.

The following is an example of what can be said in the absence of commutativity.

THEOREM 1.4: *Let rank* AB = *rank* B = r_1 *and rank* BA = *rank* A = r_2. *If* AB *and* B *are* EPr_1 *and* A *is* EPr_2 *then* BA *is* EPr_2.

PROOF: If $A\xi = 0$, then $BA\xi = 0$ and therefore $\eta(A) \subseteq \eta(BA)$. However, since rank A = rank BA, we have $\eta(A) = \eta(BA)$. Similarly, $\eta(B) = \eta(AB)$.

Then $BA\xi = 0$

$\Leftrightarrow$ $A\xi = 0$

$\Leftrightarrow$ $A^*\xi = 0$ since A is EPr_2

$\Leftrightarrow$ $B^*A^*\xi = 0$

$\Leftrightarrow$ $AB\xi = 0$ since AB is EPr_2

$\Leftrightarrow$ $B\xi = 0$

$\Leftrightarrow$ $B^*\xi = 0$

$\Leftrightarrow$ $A^*B^*\xi = 0$.

Hence $\eta(BA) \subseteq \eta(A^*B^*) = \eta((BA)^*)$. But rank BA = rank $(BA)^*$ and therefore $\eta(BA) = \eta(A^*B^*)$. Thus BA is EPr_2. We shall return to this theme again.

4. Now we are ready to consider EPr matrices over an arbitrary field F.

DEFINITION: Let F be a field and λ an involutory automorphism of F; that is λ is an automorphism of F such that λ^2 is the identity map. For $a \in F$, let $\lambda(a) = \bar{a} \in F$ and for $A = [a_{ij}]$ set $A^* = [b_{ij}]$ where $b_{ij} = \bar{a}_{ji}$. We say an $n \times n$ matrix A with entries from F is EPr if the following conditions are satisfied

(1) A has rank r,

(2) $\sum_{i=1}^{n} \alpha_i A_i = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0$ ($\alpha_i \in F$, $i = 1, \ldots, n$).

We first prove an analogue of theorem 1.1.

THEOREM 1.1′: *Let* A *be an* n × n *matrix with entries in a field* F. *Then statements* (1), (3), (5), (6), (7), (8) *and* (9) *of theorem* 1.1 *are equivalent.*

PROOF: Examination of the proof of theorem 1.1 indicates that no properties peculiar to $\mathscr{C}$ were used and all properties of the complex conjugation that were essential carry over to λ; that is $\bar{a} = 0$ if and only if $a = 0$ and $\overline{ab} = \bar{a}\,\bar{b}$, $\overline{a+b} = \bar{a} + \bar{b}$ whether $\bar{a}$ denotes the complex conjugate of $a \in \mathscr{C}$ or denotes $\lambda(a)$ for $a \in F$.

It should be noted that statements (2) and (4) of theorem 1.1 may fail to be equivalent to statement (1). For example, consider

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

over $GF(5)$. The only automorphism of $GF(5)$ is the identity (since the image of 1 is 1) and so $A^* = A^t$. A simple calculation shows that $\eta(A) = \eta(A^t)$ and a basis vector is $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, therefore, A is EP_2.

But there is no U satisfying $UU^* = I$ and such that

$$UAU^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is 2×2 and nonsingular. For the existence of such a matrix U would imply rank $A^2 =$ rank $D^2 =$ rank $D =$ rank A. However, from A^2 as calculated below, it is clear that rank $A^2 <$ rank A.

We have seen that if a complex matrix A is EPr, then A^n is EPr. This situation does not hold for an arbitrary field. In fact, for the example above, we have

$$A^2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus rank $A^2 = 1$. But A^2 is not EP_1 since

$$A^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, \; (A^2)^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Since $A^3 = A^2$, *no* higher power of A is EP.

We point out that these conditions fail here because the proof of Schur's theorem, previously used, requires that the characteristic vectors of A not be isotropic vectors under the Euclidean inner product. Over $\mathscr{C}$, no nonzero vector is isotropic. Over $GF(5)$, the vector $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is isotropic.

One should note that A has minimal polynomial $x^2(x-1)$. In fact, if x^2 does not divide the minimal polynomial of A, then rank $A^2 =$ rank A. Hence $A^2\xi = 0$ implies $A\xi = 0$ so $A^*\xi = 0$ and thus $(A^2)^*\xi = 0$. Then A^2 is EP.

On the other hand the EP_2 matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

over $GF(5)$ has minimal polynomial $x^2(x-1)$, but A^2 is not EP.

We have

THEOREM 1.3′: *Let* A *be an* EP_{r_1} *matrix and* B *an* EP_{r_2} *matrix over a field* F. *If* AB = BA *and neither* η(A) *nor* η(B) *contains isotropic vectors, then* AB *is* EPr.

PROOF: The essential part of theorem 1.2 (from which theorem 1.3 follows immediately) which requires the complex field is the equality

$$\eta(A) \cap \eta(A)^{\perp} = (0) = \eta(B) \cap \eta(B)^{\perp}$$

But, under the hypothesis of theorem 1.3′, this holds.

It is of interest to have a theorem along the lines of theorem 1.3 which places no restrictions on F. We now give such a theorem, which will also show that any example of the type given at the beginning of this section must involve a matrix of size at least 3×3.

THEOREM 1.5: *Let* A *be an* EP_{r_1} *matrix and* B *an* EP_{r_2} *matrix over a field* F *such that* AB = BA. *If* r = *rank* AB = $\min(r_1, r_2)$, *then* AB *is* EPr.

PROOF: Suppose that $r = r_1$. If $A\xi = 0$ where $\xi \epsilon F_n$, then $0 = BA\xi = AB\xi$ so that $\eta(A) \subseteq \eta(AB)$.

Since $r=r_1$, it follows that $\eta(A)=\eta(AB)$. Thus, if $AB\xi=0$ then

$$A\xi=0,$$
$$A^*\xi=0,$$
$$B^*A^*\xi=0$$

and so

$$\eta(AB)\subseteq\eta(B^*A^*).$$

But rank $AB=$ rank $(AB)^*$, so $\eta(AB)=\eta(B^*A^*)$ and therefore, AB is EPr_1.

The argument is similar if $r=r_2$.

COROLLARY: *If* A *is a* 2×2 EPr_1 *matrix over* F *and rank* $\mathrm{A}^2=\mathrm{r}_2$ *then* A^2 *is* EPr_2.

PROOF: *Case 1.* $r_1=2$. Then A^2 is nonsingular, and hence is EP_2.

Case 2. $r_1=1$. If $r_2=1$, apply theorem 1.5. If $r_2=0$, then $A^2=0$ and the result is trivial.

Case 3. $r_1=0$. Trivial.

We note that theorem 1.4 remains valid for matrices with entries from an arbitrary field, and then show

THEOREM 1.6: *Let* A, B, *and* AB *be* EPr *matrices over a field* F. *Then* BA *is an* EPr *matrix.*

PROOF: It follows from theorem 1.4 that it suffices to prove $r=$ rank BA.

In view of the equality $(W^\perp)^\perp=W$ for subspaces $W\subseteq F_n$ it follows that the formulas

$$(\alpha)\qquad \text{rank } AB=r-\dim(\eta(A)\cap\eta(B^*)^\perp),$$
$$(\beta)\qquad \text{rank } BA=r-\dim(\eta(B)\cap\eta(A^*)^\perp)$$

(see [6], theorem 7.8) are valid. From (α) and the hypothesis that B is EPr we have

$$r=r-\dim(\eta(A)\cap\eta(B)^\perp)$$

so

$$\eta(A)\cap\eta(B)^\perp=(0)$$

and thus $F_n=\eta(A)\oplus\eta(B)^\perp$ (by a dimension argument). Hence

$$(0)=(F_n)^\perp=(\eta(A)\oplus\eta(B)^\perp)^\perp\supseteq\eta(A)^\perp\cap\eta(B)^{\perp\perp}=\eta(A)^\perp\cap\eta(B)$$

so that $\dim(\eta(B)\cap\eta(A)^\perp)=0$. Then $r=$ rank BA.

THEOREM 1.7: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over a field* F. *Then we may write*[6]

$$A^*=AN,\ B^*=BM,$$

for some nonsingular matrices N *and* M. *If* AB = BA, *then* AB *is* EPr *if and only if* $\mathrm{N}\eta(\mathrm{AB})\subseteq\eta(\mathrm{AB})$.

PROOF: Since $B^*=M^{-*}B$ it follows that

$$B^*A^*=M^{-*}BAN=M^{-*}ABN.$$

Suppose AB is EPr and $AB\xi=0$. Then

$$0=B^*A^*\xi=M^{-*}ABN\xi=AB(N\xi)$$

which implies that $N\xi\epsilon\eta(AB)$.

[6] The following matrix N would be $(N^*)^{-1}$ in the notation of (6) of Theorem 1.1.

Conversely, if $N\eta(AB) \subseteq \eta(AB)$, then by repeating the above argument we can show that AB is EPr.

THEOREM 1.8: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over* F. *Let* $\mathrm{AB}=\mathrm{BA}$ *and let* $\mathrm{A}^*=\mathrm{AN}$, *where* N *is nonsingular. If* $\eta(\mathrm{A})+\eta(\mathrm{B})=\eta(\mathrm{AB})$, *then* $\mathrm{N}\eta(\mathrm{AB}) \subseteq \eta(\mathrm{AB})$.

PROOF: Since A is EPr_1 then $\eta(A^*)=\eta(A)$ and therefore

$$N\eta(A) \subseteq \eta(A) \subseteq \eta(AB).$$

Hence it is sufficient to show that $N\eta(B) \subseteq \eta(AB)$.

First we note that $A^*\eta(B) \subseteq \eta(B)$. For if $x \epsilon \eta(B)=\eta(B^*)$, then

$$B^*A^*x=A^*B^*x=0.$$

Moreover since B is EPr_2 it follows that $BA^*x=0$ and hence $A^*x \epsilon \eta(B)$.

Again let $x \epsilon \eta(B)$. Then

$$ABNx=BANx=BA^*x=0$$

as required, completing the proof.

The converse of theorem 1.7 is false, as the following shows. Let p be a prime and let $F=GF(p)$. Let $A=B$ be a $p \times p$ matrix over F with all entries equal to 1. Then $A=B$ is EP_1 and $\eta(A)+\eta(B)=\eta(A)$. But $A^2=0$ so $\eta(AB)=\eta(A^2)=(GF(p))_n \neq \eta(A)+\eta(B)=\eta(A)$, since dim $\eta(A)=p-1$. However, $A=A'$ so the matrix N of theorem 1.7 may be chosen to be I.

We close this section by noting that theorems 1.6 and 1.7 have analogs for the matrix M.

3. A^* as Unitary Multiple of A

1. It is known that if A is a complex normal matrix then

$$A^*=NA=AN$$

where N satisfies $NN^*=I$, that is, N is unitary. By condition 6 of theorem 1.1, A is an EPr matrix. The matrix

$$\mathrm{A}=\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

with entries from $GF(5)$ satisfies $AA^*=A^*A$ (where the automorphism used to define A^* is the identity). But

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix}=0,\ A^*\begin{pmatrix} 1 \\ 2 \end{pmatrix}=\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

so that A is not EPr.

Thus in order to be able to obtain a relationship of the form (1) it is necessary that A be both normal and EPr. The conjecture was made that these conditions are also sufficient. The main purposes of this section are to show that

(1) with some added hypotheses a relationship of the form (1) exists, but

(2) without any added hypotheses, the conjecture is not universally true.

The class of normal *EPr* matrices having such a relationship will be extended to zero-type[7] *EPr* matrices over a field of characteristic $\neq 2$ in subsection 2. Also, an example will be given of a 4×4 zero-type EP_2 matrix over an arbitrary field of characteristic 2 which does not have a relationship of type (1).

2. We begin by finding sufficient conditions for a relationship of type (1).

THEOREM 2.1 ([10]): *if* A *is normal and has the same rank as* AA*, *then* A *is* EPr.

Over the complex field an *EPr* matrix is not necessarily normal and over an arbitrary field the addition of the hypothesis rank $A=$ rank AA^* does not imply normality. For example

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

considered over $\mathscr{C}$ or over $GF(3)$ is EP_2 and satisfies rank $A=$ rank AA^*. But A is not normal.

THEOREM 2.2 ([10]): *Let* A *be an* n × n *matrix such that rank* A = *rank* AA*. *Then* A *is normal if and only if* A* = NA = AN *where* N *satisfies* NN* = I.

THEOREM 2.3 ([11]): *Let* A *be an* n × n *matrix of rank* r. *Then* A *is a normal* EPr *matrix if and only if there is a nonsingular matrix* M *such that* A* = AM = MA.

We do not repeat the proofs here. The essence of the argument will be repeated in the proof of theorem 2.4.

The main object of this section is to improve theorems 2.2 and 2.3, that is, to remove the restriction that A and AA^* have the same rank and, at the same time, obtain a matrix N such that $A^*=NA=AN$ and $NN^*=I$.

LEMMA: *Let* A *be an* EPr *matrix. We may express* A *as*[8]

$$\mathrm{A}=\mathrm{T}\left[\begin{array}{c|c} \mathrm{I} & \mathrm{0} \\ \hline \mathrm{X} & \mathrm{I} \end{array}\right]\left[\begin{array}{c|c} \mathrm{D} & \mathrm{0} \\ \hline \mathrm{0} & \mathrm{0} \end{array}\right]\left[\begin{array}{c|c} \mathrm{I} & \mathrm{X^*} \\ \hline \mathrm{0} & \mathrm{I} \end{array}\right]\mathrm{T^*}. \tag{2}$$

Then A *is normal if and only if*

$$\mathrm{D(I+X^*X)D^*=D^*(I+X^*X)D}. \tag{3}$$

PROOF: Suppose A is normal. Then

$$AA^*=T\left[\begin{array}{c|c} D(I+X^*X)D^* & D(I+X^*X)D^*X^* \\ \hline XD(I+X^*X)D^* & XD(I+X^*X)D^*X^* \end{array}\right]T^*$$

and

$$A^*A=T\left[\begin{array}{c|c} D^*(I+X^*X)D & D^*(I+X^*X)DX^* \\ \hline XD^*(I+X^*X)D & XD^*(I+X^*X)DX^* \end{array}\right]T^*.$$

Comparing the (1, 1) positions of $AA^*=A^*A$ shows that (3) holds.

Conversely, if

$$D(I+X^*X)D^*=D^*(I+X^*X)D,$$

[7] See above Theorem 2.5 for definition.

[8] Here T is a permutation matrix corresponding to the P in (7) of Theorem 1.1.

then

$$D(I+X^*X)D^*X^* = D^*(I+X^*X)DX^*,$$

$$XD(I+X^*X)D^* = XD^*(I+X^*X)D,$$

and

$$XD(I+X^*X)D^*X^* = XD^*(I+X^*X)DX^*$$

Hence, corresponding blocks of AA^* and A^*A are equal so that A is normal.

THEOREM 2.4: *Let* A *be a normal* EPr *matrix with* A *expressed as in* (2); *a necessary and sufficient condition that there exist a unitary* N *such that* $A^* = AN = NA$ *is that there exist an* $(n-r)\times(n-r)$ *matrix* G_0 *such that*

$$-I + XDD^{-*}D^{-1}D^*X^* - XDD^{-*}X^*G_0 - G_0^*XD^{-1}D^*X^* + G_0^*(I+XX^*)G_0 = 0. \tag{4}$$

PROOF: Suppose that $A^* = NA$. Since T is nonsingular we may write N as follows:

$$N = T\left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right] T^* \tag{5}$$

where B is $r\times r$, C is $r\times(n-r)$, F is $(n-r)\times r$ and G is $(n-r)\times(n-r)$. Substituting (5) into $A^* = NA$ and simplifying we obtain

$$T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right] T^* = T\left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right] T^* T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right] T^*,$$

$$\left[\begin{array}{c|c} D^* & 0 \\ \hline XD^* & 0 \end{array}\right] = \left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline XD & 0 \end{array}\right] = \left[\begin{array}{c|c} BD+CXD & 0 \\ \hline FD+GXD & 0 \end{array}\right].$$

Thus,

$$D^* = BD + CXD$$

so that

$$B = D^*D^{-1} - CX.$$

Also,

$$XD^* = FD + GXD.$$

Hence

$$F = XD^*D^{-1} - GX.$$

Thus the general solution of $A^* = NA$ is given by

$$N = T\left[\begin{array}{c|c} D^*D^{-1} - CX & C \\ \hline XD^*D^{-1} - GX & G \end{array}\right] T^*.$$

We further require that the matrix N obtained above satisfy $A^* = AN$. Using the representations (2) for A (and A^*) it follows that

$$T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right] T^*$$

$$= T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right] T^* T\left[\begin{array}{c|c} D^*D^{-1}-CX & C \\ \hline XD^*D^{-1}-GX & G \end{array}\right] T^*,$$

$$\left[\begin{array}{c|c} D^* & D^*X^* \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} D & DX^* \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} D^*D^{-1}-CX & C \\ \hline XD^*D^{-1}-GX & G \end{array}\right]$$

$$= \left[\begin{array}{c|c} DD^*D^{-1}-DCX+DX^*XD^*D^{-1}-DX^*GX & DC+DX^*G \\ \hline 0 & 0 \end{array}\right]$$

Hence,

$$D^* = DD^*D^{-1} - DCX + DX^*XD^*D^{-1} - DX^*GX = D(I + X^*X)D^*D^{-1} - D(C + X^*G)X \tag{6}$$

and

$$D^*X^* = DC + DX^*G$$

$$C = D^{-1}D^*X^* - X^*G. \tag{7}$$

Substituting (7) in (6) we have

$$D^* = D(I + X^*X)D^*D^{-1} - D(D^{-1}D^*X^* - X^*G + X^*G)X = D(I + X^*X)D^*D^{-1} - D^*X^*X$$

which is equivalent to (3). By the lemma it follows that

$$N = T\left[\begin{array}{c|c} D^*D^{-1} - D^{-1}D^*X^*X + X^*GX & D^{-1}D^*X^* - X^*G \\ \hline XD^*D^{-1} - GX & G \end{array}\right] T^* \tag{8}$$

satisfies $A^* = NA = AN$.

Finally, we require that G be chosen so that $N^*N = I$. By computing N^* from (8) and then multiplying out N^*N we obtain

$$N^*N = T\left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] T^*$$

where

$$\begin{aligned} P &= D^{-*}DD^*D^{-1} - D^{-*}DD^{-1}D^*X^*X + D^{-*}DX^*GX - X^*XDD^{-*}D^*D^{-1} \\ &\quad + X^*XDD^{-*}D^{-1}D^*X^*X - X^*XDD^{-*}X^*GX + X^*G^*XD^*D^{-1} - X^*G^*XD^{-1}D^*X^*X \\ &\quad + X^*G^*XX^*GX + D^{-*}DX^*XD^*D^{-1} - D^{-*}DX^*GX - X^*G^*XD^*D^{-1} + X^*G^*GX \\ &= I_r - X^*[I_{n-r} - XDD^{-*}D^{-1}D^*X^* + XDD^{-*}X^*G + G^*XD^{-1}D^*X^* - G^*(I + XX^*)G]X. \end{aligned} \tag{9}$$

$$\begin{aligned} Q &= D^{-*}DD^{-1}D^*X^* - D^{-*}DX^*G - X^*XDD^{-*}D^{-1}D^*X^* + X^*XDD^{-*}X^*G \\ &\quad + X^*G^*XD^{-1}D^*X^* - X^*G^*XX^*G + D^{-*}DX^*G - X^*G^*G \\ &= X^*[I - XDD^{-*}D^{-1}D^*X^* + XDD^{-*}X^*G + G^*XD^{-1}D^*X^* - G^*(I + XX^*)G]. \end{aligned} \tag{10}$$

$$R = Q^*. \tag{11}$$

$$S = XDD^{-*}D^{-1}D^*X^* - XDD^{-*}X^*G - G^*XD^{-1}D^*X^* + G^*XX^*G + G^*G$$

$$= XDD^{-*}D^{-1}D^*X^* - XDD^{-*}X^*G - G^*XD^{-1}D^*X^* + G^*(I + XX^*)G = S^*. \tag{12}$$

Hence

$$P = I_r - X^*(I_{n-r} - S)X,$$

$$Q = X^*(I_{n-r} - S),$$

$$R = (I_{n-r} - S)^*X = (I_{n-r} - S)X.$$

If for some G we have $N^*N = I$, then $S = I_{n-r}$ and hence (12) reduces to (4) as required.

Conversely, suppose there is a G_0 satisfying (4). If this value is substituted for G in (8) then (12) reduces to $I_{n-r} - S = 0$. Hence

$$P = I_r - 0 = I_r,$$

$$Q = 0,$$

$$R = 0$$

so that $N^*N = I$ as required completing the proof.

In ([10], p. 3) we have the added hypothesis that rank A = rank AA^*. From this it follows as in [10] that $I + X^*X$ is nonsingular. Theorem 2.2 may be established by verifying that

$$G_0 = X(I + X^*X)^{-1}(D^{-1}D^* - I)X^* + I$$

is a solution of (4).

DEFINITION: Let $n = 2r$ (r is a positive integer). An $n \times n$ ***EPr*** matrix A such that $AA^* = 0$, is called a zero-type matrix; that is the columns of A are pairwise orthogonal and individually isotropic with respect to the inner product induced on F_n by λ.)

THEOREM 2.5: *A zero-type matrix is normal.*

PROOF: Since A is ***EPr***

$$A = T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_r \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_r \end{array}\right]T^*$$

where X is $r \times r$, so that

$$0 = AA^* = \left[\begin{array}{c|c} D(I + X^*X)D^* & 0 \\ \hline 0 & 0 \end{array}\right].$$

Since D is nonsingular, it follows that $I + X^*X = 0$. By the lemma before Theorem 2.4, A is normal.

The preceding proof did not use the fact that $n = 2r$. Nor does the following alternate proof: $AA^* = 0$ implies that each column of A^* lies in $\eta(A)$. Since A is EPr, it follows that $A^*A^* = 0$ and thus that $AA = 0$. Hence each column of A is in $\eta(A)$, and since A is EPr, $A^*A = 0$.

A similar proof shows that EPr matrix A is normal if $A^2 = 0$.[9]

[9] We are indebted to A. J. Goldman (National Bureau of Standards) for the observations of the preceeding two paragraphs.

COROLLARY: *If* A *is a zero-type matrix, then the square matrix* X *of theorem* 1.1, *part* (7) *satisfies*

$$I+X^*X=I+XX^*=0.$$

THEOREM 2.6: *If* A *is a zero-type matrix over a field of characteristic* $\neq 2$, *then there is a unitary matrix* N *such that* $A^*=NA=AN$.

PROOF: According to theorem 2.5 it suffices to find an $(n-r)\times(n-r)$ matrix G_0 satisfying (4). However, by the above corollary, for a zero-type matrix, $I+XX^*=0$ so that (4) reduces to

$$\begin{aligned}0&=-I+XDD^{-*}D^{-1}D^*X^*-XDD^{-*}X^*G_0-G_0^*XD^{-1}D^*X^*\\&=-I+XDD^{-*}D^{-1}D^*X^*-XDD^{-*}X^*G_0-(XDD^{-*}X^*G_0)^*.\end{aligned}$$

Set $Y=XDD^{-*}X^*G_0$. Then (4) becomes

$$0=-I+XDD^{-*}D^{-1}D^*X^*-Y-Y^*. \tag{13}$$

Set $H=-\frac{1}{2}(I-XDD^{-*}D^{-1}D^*X^*)$. Then $H=H^*$ and $Y=H$ is a solution of (13). Thus

$$G_0=(XDD^{-*}X^*)^{-1}H=-\frac{1}{2}(XDD^{-*}X^*)^{-1}(I-XDD^{-*}D^{-1}D^*X^*)$$

satisfies (4), completing the proof.

We now give an example to show that the requirement characteristic $\neq 2$ is necessary. Let

$$A=\begin{bmatrix}1&0&1&0\\1&1&1&1\\1&0&1&0\\1&1&1&1\end{bmatrix}$$

over a field of characteristic 2. Then

$$A=\left[\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline1&0&1&0\\0&1&0&1\end{array}\right]\quad\left[\begin{array}{cc|cc}1&0&0&0\\1&1&0&0\\\hline0&0&0&0\\0&0&0&0\end{array}\right]\quad\left[\begin{array}{cc|cc}1&0&1&0\\0&1&0&1\\\hline0&0&1&0\\0&0&0&1\end{array}\right]$$

so that A is EP_2. Moreover, $AA^*=0$ so that A is a zero-type matrix. Equation (13) of theorem 2.6 reduces to

$$Y+Y^*=\begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

Thus, the impossibility in some cases of finding a unitary matrix N such that $A^*=AN=NA$ follows from the following theorem and theorem 2.5.

THEOREM 2.7: *Let* F *be a field of characteristic* 2 *and* λ *the identity. If* S *is a symmetric matrix over* F, *then a necessary and sufficient condition that there exist a matrix* R *such that* $R+R^*=S$ *is that* S *have only zeros on the diagonal.*

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PROOF: Suppose $R+R^*=S$. Because $\alpha+\alpha=0$ for $\alpha\epsilon F$, it is clear that the diagonal elements of S are zero.

Conversely, let the diagonal elements of S be zeros. Let R have zeros on and below the diagonal and be identical with S elsewhere. Then $R+R^*=S$, completing the proof.

A matrix M satisfying the conditions of theorem 2.3 is

$$M=\begin{bmatrix}1&1&1&0\\1&0&0&1\\0&0&0&1\\0&0&1&1\end{bmatrix}$$

It would be desirable to relax the requirement $n=2r$ in the definition of zero-type matrix. Such a change, however, leads to difficulties in the ensuing matrix equation which we cannot presently handle. In particular, the condition $I+X^*X=0$ (which continues to hold) need not imply $I+XX^*=0$, and so (4) retains its formidable character. For example, consider

$$A=A^*=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

over a field of characteristic 3. Then rank $A=1$ and $AA^*=0$. Suppose $X=(a,\ b)^t$ and $D=[d]$ in (7) of Theorem 1.1. For any permutation matrix T,

$$A=T^{-1}AT=\begin{bmatrix}d&d\bar{a}&d\bar{b}\\da&da\bar{a}&da\bar{b}\\db&db\bar{a}&db\bar{b}\end{bmatrix}$$

Hence $X=(1,\ 1)^t$, so that

$$I+XX^*=\begin{bmatrix}2&1\\1&2\end{bmatrix}\neq 0$$

although $I+X^*X=0$. Note that $N=I$ is a unitary solution of $A^*=AN=NA$.

3. It was proved by Williamson [24] that any complex normal matrix A may be written as a polynomial in A^* with complex coefficients. An example is given in [10] to show that this is not necessarily the case for a normal matrix over an arbitrary field F. That example gives a normal but not EPr matrix. Instead of repeating this example we give a matrix A which is normal and EPr, yet A^* is not a polynomial in A. The matrix A of the example before theorem 2.7 suffices. $A^2=0$ so we try to find α and β in $GF(2)$ such that $A^*=\alpha A+\beta I$.

$$\begin{bmatrix}1&1&1&1\\0&1&0&1\\1&1&1&1\\0&1&0&1\end{bmatrix}=\alpha\begin{bmatrix}1&0&1&0\\1&1&1&1\\1&0&1&0\\1&1&1&1\end{bmatrix}+\beta\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}\alpha+\beta&0&\alpha&0\\\alpha&\alpha+\beta&\alpha&\alpha\\\alpha&0&\alpha+\beta&0\\\alpha&\alpha&\alpha&\alpha+\beta\end{bmatrix}$$

so that comparing the (1, 3) and (2, 1) positions leads to

$$\alpha=1,\ \alpha=0.$$

Hence A^* is not a polynomial in A.

We close this section by stating the only known result concerning the expression of A^* as a polynomial in A and giving an example.

THEOREM 2.8 ([10]): *Let* A *be an* n × n *matrix over a field* F *and let* K *be a field containing* F *and the characteristic roots of* A. *If rank* $(A-\beta I)(A-\beta I)^*=$ *rank* $(A-\beta I)=r_\beta$ *and* $A-\beta I$ *is* EPr_β *for each* $\beta\epsilon K$, *then* A *is normal and* A^* *may be written as a polynomial in* A.

However, the hypotheses are not also necessary. For example

$$A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

over $GF(2)$ satisfies the conclusion of the theorem, but rank $A \neq$ rank AA^*. Moreover, this example is easily adapted to $GF(p)$, $p=$ prime.

4. *EP* Linear Transformations

1. In this section we introduce the notion of an *EP* linear transformation acting on a finite-dimensional vector space. We develop a theory for *EP* linear transformations and then show how some of the results of the previous sections may be derived by means of this new concept.

Finally, motivated by the well-known notion of normal linear transformation, we introduce the concept of θ-normality and apply it to the study of normal *EPr* matrices.

2. Notation: Throughout, V denotes an n-dimensional vector space over a field F, $V_n(F)$ [10] denotes the vector space of $m\times 1$ column vectors with components from F, $\hat{V}$ denotes the dual space of V (that is, the space of linear transformations from V to F), $L(V)$ denotes the space of linear transformations from V to V, and for $T\epsilon L(V)$, $\eta(T)$ and $R(T)$ denote the null space and range space of T respectively. $[T]_B$ denotes the matrix of T relative to the basis B of V. $\eta(A)$ and $R(A)$ denote the null space and range space of the $n\times n$ matrix A respectively.

We recall the essential facts concerning $T^*\epsilon L(\hat{V})$, the linear transformation dual to T. For $x\epsilon V$ and $\hat{y}\epsilon\hat{V}$ we set

$$T^*\hat{y}(x)=\overline{\hat{y}(Tx)},$$

where $\alpha\to\bar{\alpha}$ is an involution of F. Generally one sets $T^*\hat{y}(x)=\hat{y}(Tx)$, but our special purposes require the conjugation and no great change ensues. This gives a well-defined linear transformation on $\hat{V}$. Moreover, if $v_1, \ldots, v_n$ and $\hat{v}_1, \ldots, \hat{v}_n$ are dual bases, that is, $\hat{v}_i(v_j)=\delta_{ij}$, then the matrices of T relative to $v_1, \ldots, v_n$ and of T^* relative to $\hat{v}_1, \ldots, \hat{v}_n$ are conjugate transposes. These elementary facts will suffice for our purposes. For further details see the lectures by N. Jacobson [8, pp. 51–60].

DEFINITION: Let $T\epsilon L(V)$. T is an *EP* linear transformation if there is a basis $u_1, \ldots, u_n$ of V and a dual basis $\hat{u}_1, \ldots, \hat{u}_n$ of $\hat{V}$ such that

$$T\left(\sum_{i=1}^{n}\alpha_i u_i\right)=0 \text{ if and only if } T^*\left(\sum_{i=1}^{n}\alpha_i\hat{u}_i\right)=0.$$

Dual bases satisfying this requirement will be called special bases and the basis $u_1, \ldots, u_n$ will be called a special basis. (So if reference is made to a special basis, it is to be understood that there is a dual basis such that together these dual bases form special bases.)

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[10] Previously denoted F_n.

THEOREM 3.1: *Let* TϵL(V). T *is* EP *with special basis* B: $b_1, \ldots, b_n$ *if and only if* $[T]_B$ *is an* EP *matrix.*

PROOF: Consider the isomorphism f_B: $V \to f_B(V) = V_n(F)$ defined by

$$f_B\left(\sum_{i=1}^{n} \alpha_i b_i\right) = (\alpha_1, \ldots, \alpha_n)'. \tag{1}$$

Note that $E: f_B(b_1), \ldots, f_B(b_n)$ is the natural basis of $V_n(F)$. f_B induces an isomorphism $F_B: L(V) \to F_B(L(V)) = L(V_n(F))$ defined by

$$(F_B T)(f_B x) = f_B(Tx) \qquad (T\epsilon L(V),\ x\epsilon V).$$

Note that

$$[F_B T]_E = [T]_B. \tag{2}$$

If $\hat{B}: \hat{b}_1, \ldots, \hat{b}_n$ is the dual basis to B, then there is also an isomorphism $\hat{f}_B: \hat{V} \to \hat{f}_B(\hat{V}) = V_n(F)$ defined by

$$\hat{f}_B\left(\sum_{i=1}^{n} \beta_i \hat{b}_i\right) = (\beta_1, \ldots, \beta_n)'$$

and $\hat{f}_B(\hat{b}_1), \ldots, \hat{f}_B(\hat{b}_n)$ is again the natural basis of $V_n(F)$. $\hat{f}_B$ induces an isomorphism $\hat{F}_B: L(\hat{V}) \to \hat{F}_B(L(\hat{V})) = L(\hat{V}_n(F))$ defined by

$$(\hat{F}_B \hat{T})(\hat{f}_B \hat{x}) = \hat{f}_B(\hat{T}\hat{x}) \qquad (\hat{T}\epsilon L(\hat{V}),\ \hat{x}\epsilon\hat{V}).$$

Note that

$$[\hat{F}_B \hat{T}]_E = [\hat{T}]_{\hat{B}}.$$

As remarked earlier, for any $T\epsilon L(V)$,

$$[T^*]_{\hat{B}} = [T]_B^*.$$

First assume T is *EP* with B as special basis. Then

$$0 = [T]_B f_B(x) = [F_B T]_E f_B(x) = (F_B T)(f_B(x)) = f_B(Tx)$$

implies that

$$0 = Tx = T\left(\sum_{i=1}^{n} \alpha_i b_i\right)$$

and thus that

$$0 = T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = T^*\hat{x}.$$

This in turn implies

$$0 = \hat{f}_B(T^*x) = (\hat{F}_B T^*)\hat{f}_B(\hat{x}) = [\hat{F}_B T^*]_E \hat{f}_B(\hat{x}) = [T^*]_{\hat{B}} \hat{f}_B(\hat{x}) = [T^*]_{\hat{B}}(\alpha_1, \ldots, \alpha_n)' = [T]_B^* f_B(x).$$

Furthermore, the resoning is reversible. Thus for all $\xi = f_B(x) \epsilon V_n(F)$ we have

$$[T]_B \xi = 0 \quad \text{iff} \quad [T]_B^* \xi = 0,$$

and so $[T]_B$ is an EP matrix.

Next assume $[T]_B$ is an EP matrix. Then

$$Tx = T\left(\sum_{i=1}^{n} \alpha_i b_i\right) = 0 \cdot$$

implies

$$0 = f_B(Tx) = (F_B T)(f_B(x)) = [F_B T]_E f_B(x) = [T]_B f_B(x),$$

which in turn implies

$$0 = [T]_B^* f_B(x) = [T^*]_{\hat{B}}(\alpha_1, \ldots, \alpha_n)^t = [\hat{F}_B T^*]_E \hat{f}_B \left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right)$$

$$= (\hat{F}_B T^*) \hat{f}_B \left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = \hat{f}_B\left(T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right)\right).$$

Thus $T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = 0$. Furthermore, the reasoning is reversible and so T is an EP transformation with B as special base.[11]

If there exist special bases for T it is not necessarily true that any pair of dual bases provides special bases. For example, consider the linear transformation T whose matrix relative to the natural basis e_1, e_2, e_3 of $V_3(F)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the dual basis. Then $Te_2 = 0$. Also

$$T^* \hat{e}_2(e_1) = \overline{\hat{e}_2(Te_1)} = \overline{\hat{e}_2(e_1 + e_3)} = 0,$$

$$T^* \hat{e}_2(e_2) = \overline{\hat{e}_2(Te_2)} = \overline{\hat{e}_2(0)} = 0,$$

$$T^* \hat{e}_2(e_3) = \overline{\hat{e}_2(Te_3)} = \overline{\hat{e}_2(e_3)} = 0,$$

so that $T^* e_2 = 0$.

Suppose $T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right) = 0$. Then

$$0 = T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right)(e_1) = \overline{\sum_{i=1}^{3} \alpha_i e_i(e_1 + e_3)} = \overline{\alpha_1} + \overline{\alpha_3},$$

[11] We are indebted to A. J. Goldman (National Bureau of Standards) for simplifying an earlier version of this result and bringing the proof to its present simpler form.

$$0 = T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right)(e_2) = \overline{\sum_{i=1}^{3} \alpha_i e_i(0)} = 0,$$

$$0 = T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right)(e_3) = \overline{\sum_{i=1}^{3} \alpha_i e_i(e_3)} = \overline{\alpha_3},$$

and hence $\alpha_1 = \alpha_3 = 0$ so that $\sum_{i=1}^{3} \alpha_i e_i = \alpha_2 e_2$. But $T(\alpha_2 e_2) = 0$. Thus T is EP with the natural basis and its dual as special bases.

However, consider the following dual bases. $v_1 = e_1$, $v_2 = e_1 + e_2$, $v_3 = e_3$ and $v_1 = e_1 - e_2$, $v_2 = e_2$, $v_3 = e_3$. $T(v_1 - v_2) = 0$. But

$$T^*(v_1 - v_2)(v_1) = \overline{(v_1 - v_2)(Tv_1)} = \overline{(e_1 - 2e_2)(e_1 + e_3)} = 1 \neq 0$$

so that $T^*(v_1 - v_2) \neq 0$. Hence these dual bases do not satisfy the requirements of the definition of special bases.

It is possible that every pair of dual bases be special bases. For example, this is the case whenever T is nonsingular.

Now we give an example of an EP linear transformation T with exactly one pair of special bases. Let T have the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

over $GF(2)$ relative to the natural basis e_1, e_2 of $V_2(GF(2))$. By the same type of calculation performed earlier one verifies that the dual bases e_1, e_2 and $\hat{e_1}$, $\hat{e_2}$ are special bases. By theorem 3.1, determining all special bases B is equivalent to finding all 2×2 nonsingular matrices P such that $[T]_B = P^{-1}AP$ is EP_1. If

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $C = P^{-1}AP$ is EP_r, then $(x, y)' \epsilon \eta(C)$ implies

$$ax = by, \tag{3}$$

$$dx = cy \tag{4}$$

since the identity is the only automorphism of $GF(2)$, and

$$C\begin{pmatrix} x \\ y \end{pmatrix} = 0 = C^*\begin{pmatrix} x \\ y \end{pmatrix}.$$

By testing the three nonzero possibilities (1, 0),(0, 1), and (1, 1) for $(x, y)'$, and eliminating the third since P is nonsingular, we find that

$$P = I, \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are the only solutions. Hence T has exactly one pair B, $\hat{B}$ of special bases.

The proof of the next theorem requires the following well-known result.

LEMMA 1: *Let* $T \in L(V)$. *Then the following are equivalent.*

(i) rank T = rank T^2.
(ii) $\eta(T) \cap R(T) = (0)$.

THEOREM 3.2: *Let* $T \in L(V)$ *and rank* T = *rank* T^2. *Then* T *is* EP.

PROOF. Let rank $T = n - r$. Let V have a basis $u_1, \ldots, u_r, u_{r+1}, \ldots, u_n$ where the first r vectors form a basis of $\eta(T)$ and the remaining $(n-r)$ vectors form a basis of $R(T)$.

Suppose $Tu_i = \sum_{k=1}^{n} \alpha_{ki} u_k (i = 1, \ldots, n)$. Then

$$\sum_{k=1}^{r} \alpha_{ki} u_k = Tu_i - \sum_{k=r+1}^{n} \alpha_{ki} u_k. \tag{5}$$

The left-hand side of (5) is in $\eta(T)$ and the right-hand side is in $R(T)$. Thus

$$\sum_{k=1}^{r} \alpha_{ki} u_k = 0$$

and

$$Tu_i = \sum_{k=r+1}^{n} \alpha_{ki} u_k \in R(T) \, (i = 1, \ldots, n).$$

If the dual basis is $\hat{u}_1, \ldots, \hat{u}_n$, then for $i = 1, \ldots, r$,

$$(T^* \hat{u}_i)(u_j) = \overline{\hat{u}_i(Tu_j)} = \begin{cases} \overline{\hat{u}_i(0)} = 0 \text{ for } j = 1, \ldots, r \\ \overline{\hat{u}_i \left(\sum_{k=r+1}^{n} \alpha_{kj} u_k \right)} = 0 \text{ for } j = r+1, \ldots, n, \end{cases}$$

so that $T^* \hat{u}_i = 0$. From rank T = rank T^* [8, p. 59] it follows that $\hat{u}_1, \ldots, \hat{u}_r$ form a basis for $\eta(T^*)$. Hence T is *EP*, completing the proof.

COROLLARY 1: *Let* $T \in L(V)$. *There is a positive integer* k *such that* T^k *is* EP.

PROOF. The sequence of subspaces

$$\eta(T) \subseteq \eta(T^2) \subseteq \ldots$$

terminates. That is, there is a positive integer k such that rank T^k = rank T^{k+1}. Thus rank T^k = rank T^{2k}.

COROLLARY 2: *If* A *is an* $n \times n$ *matrix such that rank* A = *rank* A^2, *then there is a nonsingular matrix* P *such that* $P^{-1}AP$ *is* EPr.

PROOF. If T is the linear transformation whose matrix relative to the natural basis is A, then T is *EP* and so has a special basis B. By theorem 3.1,

$$[T]_B = P^{-1}AP$$

is *EPr* where P is the matrix changing coordinates from the natural basis to the special basis.

COROLLARY 3: *Let* A *be as in corollary* 2. *Then any linear transformation* T *whose matrix relative to some basis is* A *is* EP.

REMARK. The condition of theorem 3.2 is not necessary that T be EP. For example, consider the matrix

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ over } GF(2).$$

A is EPr, since it is symmetric, so the linear transformation T of theorem 3.1 is EP. However, rank $T=1$ and rank $T^2=0$.

Furthermore, this type of example can be constructed for any nonzero characteristic.

The situation is better over the complex field.

THEOREM 3.3: *The linear transformation* T *on a vector space* V *over the complex field is* EP *if and only if rank* $T=$ *rank* T^2.

PROOF: We may assume that T is singular. Suppose that T is EP and has a special basis B. $[T]_B$ is EPr so applying theorem 1.1,

$$P[\ T\ ]_BP^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $PP^*=I$ and D is nonsingular. Since D^2 is nonsingular, rank $T^2=$ rank $P[T^2]_BP^*=$ rank T.

The converse is included in theorem 3.2.

3. DEFINITION. Let $T\epsilon L(V)$ and V and $\hat{V}$ have dual bases $\hat{B}:\hat{u}_1,\ .\ .\ .,\ \hat{u}_n$ and $\hat{B}:\ \hat{u}_1,\ .\ .\ .,\ \hat{u}_n$. Let θ_B denote the linear transformation of V onto $\hat{V}$ given by

$$\theta_B\left(\sum_{i=1}^{n}\alpha_iu_i\right)=\sum_{i=1}^{n}\alpha_i\hat{u}_i.$$

Then $\theta_B^{-1}T^*\theta_B\epsilon L(V)$. We say that T is θ_B- *normal* if T and $\theta_B^{-1}T^*\theta_B$ commute.

Suppose $Tu_i=\sum_{k=1}^{n}t_{ki}u_k(i=1,\ .\ .\ .,\ n)$. Then

$$(\theta_B^{-1}T^*\theta_B)u_i=\theta_B^{-1}T^*\hat{u}_i=\theta_B^{-1}\sum_{k=1}^{n}\bar{t}_{ik}\hat{u}_k=\sum_{k=1}^{n}\bar{t}_{ik}u_k(i=1,\ .\ .\ .,\ n)$$

and so *the matrices of* T *and* $\theta_B^{-1}T^*\theta_B$ *relative to the basis* $u_1,\ .\ .\ .,\ u_n$ *are conjugate transposes.*

Let T have matrix A relative to the basis $e_1,\ .\ .\ .,\ e_n$ and let $u_i=\sum_{k=1}^{n}p_{ki}e_k(i=1,\ .\ .\ .,\ n)$, $P=[p_{ij}]$. Then

$$[T]_B=P^{-1}AP,\ [\theta_B^{-1}T^*\theta_B]_B=[T]_B^*=P^*A^*P^{-*}.$$

Hence, in matrix terms, T *is* θ_B- *normal if and only if* PAP^{-1} *and* $P^{-*}A^*P^*$ *commute.*

LEMMA 2: *Let* $T\epsilon L(V)$ *and let* θ_B *be the mapping of the preceding definition. Suppose the underlying automorphism is the identity. Then* $(\theta_B^{-1}T^*\theta_B)^*=\theta_BT\theta_B^{-1}$.

PROOF. We use the bases $\hat{B}:u_1,\ .\ .\ .,\ u_n$ and $\hat{B}:\hat{u}_1,\ .\ .\ .,\ \hat{u}_n$ employed in defining θ_B. The proof is arranged to indicate why assuming λ to be the identity is required.

Suppose $Tu_i=\sum_{k=1}^{n}\beta_{ki}u_k(i=1,\ .\ .\ .,\ n)$ so that $T^*\hat{u}_i=\sum_{k=1}^{n}\bar{\beta}_{ik}\hat{u}_k\ (i=1,\ .\ .\ .,\ n)$. If $x=\sum_{i=1}^{n}\alpha_iu_i\epsilon V$,

then

$$[(\theta_B^{-1}T^*\theta_B)^*u_i](x) = \overline{\hat{u}_i(\theta_B^{-1}T^*\theta_B x)}$$

$$= \overline{\hat{u}_i\left(\theta_B^{-1}T^*\sum_{j=1}^{n}\alpha_j\hat{u}_j\right)}$$

$$= \overline{\hat{u}_i\left(\theta_B^{-1}\sum_{j,k=1}^{n}\alpha_j\overline{\beta}_{jk}\hat{u}_k\right)}$$

$$= \overline{\hat{u}_i\left(\sum_{j,k=1}^{n}\alpha_j\overline{\beta}_{jk}u_k\right)}$$

$$= \sum_{j=1}^{n}\overline{\alpha}_j\beta_{ji}$$

and

$$(\theta_B T\theta_B^{-1}\hat{u}_i)(x) = (\theta_B T u_i)(x)$$

$$= \left(\theta_B\sum_{k=1}^{n}\beta_{ki}u_k\right)(x)$$

$$= \left(\sum_{k=1}^{n}\beta_{ki}\hat{u}_k\right)\left(\sum_{j=1}^{n}\alpha_j u_j\right)$$

$$= \sum_{j=1}^{n}\beta_{ji}\alpha_j.$$

Since λ is the identity, $\alpha_j = \overline{\alpha}_j$; thus $(\theta_B^{-1}T^*\theta_B)^*$ and $\theta_B T\theta_B^{-1}$ agree on a basis and so they are equal.

LEMMA 3: *Let* A *be an* EPr *matrix over a field* F. *Then rank* $A^2 = rank\ AA^*$.

PROOF: We have shown that $A^* = NA$ where N is nonsingular, so $AA^* = A^2N^{-*}$.

REMARK: $A = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$ shows the converse is false.

Before stating the next theorem we indicate the notation to be used. Let $T\epsilon L(V)$ and $n-r = \text{rank } T = \text{rank } T^2$. Then $\eta(T)\cap R(T) = (0)$ and hence we may choose a basis B: $u_1, \ldots, u_n$ of V such that $u_1, \ldots, u_r$ form a basis of $\eta(T)$ and $u_{r+1}, \ldots, u_n$ form a basis of $R(T)$. Moreover, T is nonsingular on $R(T)$ and thus $Tu_{r+1}, \ldots, Tu_n$ also form a basis of $R(T)$. As in theorem 3.3,

$\hat{B}$: $\hat{u}_1, \ldots, u_r, Tu_{r+1}, \ldots, Tu_n$ and $\hat{B}$: $\widehat{u_1}, \ldots, \widehat{u_r}, \widehat{(Tu_{r+1})} \ldots, \widehat{(Tu_n)}$

form a pair of special bases for T. Let

$$v_i = u_i,\ \hat{v}_i = \hat{u}_i \qquad (i = 1, \ldots, r),$$

$$v_i = Tu_i,\ \hat{v}_i = \widehat{(Tu_i)} \qquad (i = r+1, \ldots, n),$$

and

$$\theta_B\left(\sum_{i=1}^{n}\alpha_i v_i\right) = \sum_{i=1}^{n}\alpha_i\hat{v}_i.$$

Finally, suppose the underlying automorphism on F is the *identity*. Then we have the following analog of theorem 2.2.

THEOREM 3.4: *With the notation of the preceding pargraph suppose that* T *is* θ_B*-normal. Then there is a linear transformation* $R = R_{\theta} \in L(V)$ *such that*

(i) $\theta_B^{-1}T^*\theta_B = RT = TR$,

(ii) R *is unitary relative to the inner product*

$$(x, y) = \theta_B y(x), \qquad x, y \in V,$$

(*that is*, (Rx, Ry) = (x, y) *for all* x, y),

(iii) N, *the matrix of* R *relative to the basis* B, *satisfies* $NN^* = I$.

PROOF: (i) Define R as follows.

$$Ru_i = u_i \qquad (i = 1, \ldots, r)$$

$$R(Tu_i) = \theta_B^{-1}T^*\theta_B u_i \qquad (i = r+1, \ldots, n)$$

so that R is defined on all of V since $V = \eta(T) \oplus R(T)$. Since T is EP with special bases $u_1, \ldots, u_n$ and $\hat{u}_1, \ldots, \hat{u}_n$ it follows that

$$RTu_i = R0 = 0 \qquad (i = 1, \ldots, r),$$

$$\theta_B^{-1}T^*\theta_B\hat{u}_i = \theta_B^{-1}T^*\hat{u}_i = \theta_B^{-1}0 = 0 \qquad (i = 1, \ldots, r).$$

Hence $\theta_B^{-1}T^*\theta_B = RT$. Moreover,

$$TRu_i = Tu_i = 0 \qquad (i = 1, \ldots, r),$$

$$\begin{aligned} TR(Tu_i) &= T(RTu_i) \\ &= T(\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B^{-1}T^*\theta_B Tu_i \qquad \text{(by } \theta\text{-normality)} \\ &= RT(Tu_i) \end{aligned}$$

and so $TR = RT$.

(Note that $\eta(T) \cap R(T) = (0)$ is used only to insure that R can be defined on all of V.)

(ii) For $i, j = 1, \ldots, r$, $(Rv_i, Rv_j) = (Ru_i, Ru_j) = (u_i, u_j) = (v_i, v_j)$.
For $i, j = r+1, \ldots, n$,

$$\begin{aligned} (Rv_i, Rv_j) = (RTu_i, RTu_j) &= (\theta_B^{-1}T^*\theta_B u_i, \theta_B^{-1}T^*\theta_B u_j) \\ &= \theta_B\theta_B^{-1}T^*\theta_B u_j(\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B u_j(T\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B u_j(\theta_B^{-1}T^*\theta_B Tu_i) \\ &= (\theta_B^{-1}T^*\theta_B)^*\theta_B u_j(Tu_i) \\ &= \theta_B T\theta_B^{-1}\theta_B u_j(Tu_i) \qquad \text{(by lemma 2)} \end{aligned}$$

$$= \theta_B T u_j(Tu_i)$$

$$= (Tu_i, Tu_j)$$

$$= (v_i, v_j).$$

For $i = 1, \ldots, r$ and $j = r+1, \ldots, n$,

$$(v_i, v_j) = (u_i, Tu_j) = (v_i, v_j) = \theta_B v_j(v_i) = \hat{v}_j(v_i) = 0$$

and

$$(Rv_i, Rv_j) = (Ru_i, RTu_j) = (u_i, \theta_B^{-1} T^* \theta_B u_j) = \theta_B \theta_B^{-1} T^* \theta_B u_j(u_i) = \theta_B u_j(Tu_i) = \theta_B u_j(0) = 0.$$

Finally,

$$\left(\sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{n} \beta_j v_j\right) = \sum_{j=1}^{n} \beta_j \hat{v}_j \left(\sum_{i=1}^{n} \alpha_i v_i\right)$$

$$= \sum_{i=1}^{n} \beta_i \alpha_i$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i$$

$$= \sum_{i=1}^{n} \alpha_i \hat{v}_i \left(\sum_{j=1}^{n} \beta_j v_j\right)$$

$$= \left(\sum_{j=1}^{n} \beta_j v_j, \sum_{i=1}^{n} \alpha_i v_i\right)$$

so for $i = r+1, \ldots, n$ and $j = 1, \ldots, r$ it follows that

$$(v_i, v_j) = (Tu_i, u_j) = (u_j, Tu_i)$$

$$= (Ru_j, RTu_i) \qquad \text{(from the preceding case)}$$

$$= (RTu_i, Ru_j)$$

$$= (Rv_i, Rv_j).$$

Hence $(x, y) = (Rx, Ry)$ for all $x, y \in V$.

(iii) Set $Rv_i = \sum_{j=1}^{n} r_{ji} v_j$ $(i = 1, \ldots, n)$. Then

$$\delta_{ij} = \hat{v}_j(v_i) = (v_i, v_j)$$

$$= (Rv_i, Rv_j)$$

$$= \left(\sum_{\lambda=1}^{N} r_{\lambda i} v_\lambda, \sum_{\mu=1}^{N} r_{\mu j} v_\mu \right)$$

$$= \sum_{\lambda=1}^{N} r_{\lambda i} r_{\lambda j}$$

so that $N^*N = I$, completing the proof.

COROLLARY 1: *Let* A *be a normal* n × n *matrix (relative to the identity automorphism) satisfying rank* A = *rank* AA*. *Then there is a matrix* N *such that* N*N = I *and* A* = NA = AN.

PROOF: By [10] A is EPr and so it follows from lemma 3 and Theorem 3.2 that T, the linear transformation whose matrix relative to the natural basis of $V_n(F)$, E: $e_1, \ldots, e_n$ is A, is EP with $e_1, \ldots, e_n$ and $l_1, \ldots, l_n$ as special bases.

Set $\theta_E \left(\sum_{i=1}^{n} \alpha_i e_i \right) = \sum_{i=1}^{n} \alpha_i \hat{e}_i$. Since A is normal, and

$$[T]_E = A, [\theta_E^{-1} T^* \theta_E]_E = A^*,$$

it follows that T is θ_E − normal. By (i) of Theorem 3.4,

$$A^* = [\theta_E^{-1} T^* \theta_E]_E = [RT]_E = [TR]_E = NA = AN.$$

5. Some Open Problems

(1) If A is EP and normal is A^2 necessarily EP? (See corollary 1 of theorem 1.1.)

Remarks: (i) If the conclusion of theorem 2.6 is valid for all normal, EP matrices, then the answer is yes.

(ii) One should note that A^2 is EP_0 for the example given in connection with theorem 2.6.

(2) When can a matrix of rank r be expressed as a product of EPr matrices? (See corollary 2 of theorem 1.1).

(3) If A, B, and AB are EP, it does not follow that BA is EP. What additional condition will guarantee that BA is EP? In particular, A of the example preceding theorem 1.4 is not normal. Will normality suffice? (Wiegmann's theorem 20, 9 answers this in the affirmative over the complex field).

(4) If A is an EP matrix, can x^3 divide the minimal polynomial of A?

Remarks: (i) If A is a complex EPr matrix, then

$$UAU^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is an $r \times r$ nonsingular matrix and $UU^* = I$. This implies that x^2 does not divide the minimal polynomial of A.

(ii) The matrix $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

over $GF(2)$ shows that x^2 can divide the minimal polynomial if A has entries from an arbitrary field.

(5) When is the sum of EP matrices EP?

Remark: A trivial sufficient condition is the following. $A^* = NA$, $B^* = MB$, N and M nonsingular and $N = M$.

(6) Characterize the matrices A with the property that A^* is a polynomial in A.

(7) Find what additional condition a normal matrix A must satisfy so that $A = SU = US$, $S = S^*$ and $UU^* = I$.

(8) Find necessary and sufficient conditions that a linear transformation T on a finite-dimensional vector space be EP.

(9) Let T be a linear transformation on a finite-dimensional vector space V, B_1 a basis of V and $[T]B_1 = C$. If T is EP, then there is a special basis ϵ such that $[T]B_2 = A$ is an EP matrix and so C is similar to an EP matrix. Thus determining the EP linear transformations amounts to finding all matrices similar to an EP matrix. Find them.

(10) Let A be an EP matrix, B_1 and B_2 bases of a vector space, and $[T_1]_{B1}, = A$, $[T_2]_{B2} = A$. Then T_1 and T_2 are EP and $T_1 = R^{-1}T_2R$ for some R. Does R have any special properties?

(11) Prove theorems about EP linear transformations which are analogues of those about EP matrices.

Remark: Here are two easily established results about products.

(i) If S and T are linear transformations on a finite-dimensional vector space whose minimal polynomials have 0 as a simple root, and $ST = TS$, then ST is EP.

(ii) If S and T are linear transformations on a finite-dimensional vector space that are EP and commute, S and T have a common special basis, and the null spaces of S and T contain no isotropic vectors, then ST is EP.

(12) Let T_1 and T_2 be EP linear transformations on a vector space V with special bases B_1, B_2 respectively. If T_1T_2 is EP, determine a relation between a special basis of T_1T_2 and B_1, B_2.

Remark: In this connection, consider the following. Let A be an EP matrix and rank A^2 = rank A so that A^2 is also EP. If T is a linear transformation such that $[T]_B = A$, then T is EP with special basis B and $[T^2]_B = A^2$ so T^2 is EP with special basis B.

(13) Can some basis of the null space of an EP linear transformation T always be extended to a special basis of T?

Remark: This has played a distinguished role in section 4.

(14) The examples of special bases given are extreme. Find examples which are not.

(15) Characterize those EP linear transformations T(over an arbitrary field) that satisfy rank T = rank T^2.

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(Paper 70B1–167)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

The Bernstein Form of a Polynomial

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(October 27, 1965)

1. Introduction

Let P denote a polynomial with real coefficients. In this paper we devise an algorithm for determining upper and lower bounds for the set $\{P(x): 0 \leq x \leq 1\}$.

2. Some Preliminary Observations

For a polynomial P with real coefficients, it is possible, given $\epsilon > 0$, to compute, by means of arithmetical operations whose number can be readily determined, upper and lower bounds for $\{P(x): 0 \leq x \leq 1\}$ which differ from the corresponding sharp bounds by less than ϵ. This is an immediate consequence of the simple observation that, *if* $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n (n \geq 1)$ *is a polynomial with real coefficients, then, for each positive integer* m,

$$\min \{P(k/m): k = 1, 2, \ldots, m\} - \frac{1}{m} \sum_{k=1}^{n} k|a_k| \leq P(x) \leq \max \{P(k/m): k = 1, 2, \ldots, m\} + \frac{1}{m} \sum_{k=1}^{n} k|a_k| \quad (0 \leq x \leq 1),$$

which follows from the law of the mean and the inequality $|P'(x)| \leq \sum_{k=1}^{n} k|a_k| \ (0 \leq x \leq 1)$.

This procedure has the disadvantage that an extremely large number of arithmetical operations may be required to obtain moderately good bounds. In this paper we consider an algorithm which sacrifices accuracy in order to gain tractability.

3. The Bernstein Form of a Polynomial

THEOREM. *Let* $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n$ *be a polynomial of exact degree* n (≥ 0) *with real coefficients. Then*

$$\min \{b_k: k = 0, 1, \ldots, n\} \leq P(x) \leq \max \{b_k: k = 0, 1, \ldots, n\} \quad (0 \leq x \leq 1) \qquad (1)$$

where

$$b_k = \sum_{r=0}^{k} a_r \binom{k}{r} \Big/ \binom{n}{r} \qquad (k = 0, 1, \ldots, n). \qquad (2)$$

The upper (lower) bound is sharp if and only if it is equal to b_0 *or to* b_n.

Preliminary remark. It is obvious from (2) that the bounds given by (1) are always at least as good as the crude bounds $\pm \sum_{k=0}^{n} |a_k|$.

PROOF: The following representation (to be established below, but essentially due to S. N. Bernstein[1]) holds:

$$P(x) \equiv \sum_{k=0}^{n} b_k \binom{n}{k} x^k (1-x)^{n-k} \qquad (3)$$

where b_k $(k = 0, 1, \ldots, n)$ is given by (2).[2] Since $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \equiv 1$ and $\binom{n}{k} x^k (1-x)^{n-k} \geq 0$ $(0 \leq x$

[1] Bernstein, S. N., On the best approximation of continuous functions by polynomials of a given degree (Russian), Communications of the Khar'kov Mathematical Society, Series 2, **13**, 49–194 (1912).

[2] One easily sees that $(x^k(1-x)^{n-k})_{k=0}^{n}$ forms a basis for the space of all polynomials $\sum_{k=0}^{n} c_k x^k$, c_k real. Consequently, a representation (3) with real b_k's uniquely determined must hold.

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≤ 1; $k=0, 1, \ldots, n$), (3) implies (1). We refer to $P(x)$ written in the form $\sum_{k=0}^{n} b_k \binom{n}{k} x^k(1-x)^{n-k}$ as the "Bernstein form of $P(x)$." The last sum is the Bernstein polynomial of order n of every real function f defined on $[0, 1]$ for which $f(k/n)=b_k$ $(k=0, 1, \ldots, n)$.[3]

If $\max\{b_k : k=0, 1, \ldots, n\}$ is equal to b_0 or to b_n, then the upper bound given by (1) is sharp since $P(0)=b_0$ and $P(1)=b_n$. Conversely, if the upper bound given by (1) is sharp, then it is equal to b_0 or to b_n since, if $0<x<1$, either $b_0=b_1=\ldots=b_n$ or $\sum_{k=0}^{n} b_k \binom{n}{k} x^k(1-x)^{n-k} < \max\{b_k : k=0, 1, \ldots, n\}$.

The sharpness of the lower bound can be treated in an analogous fashion.

In order to establish (3), we observe that

$$\sum_{r=0}^{n} a_r x^r \equiv \sum_{r=0}^{n} a_r x^r \{x+(1-x)\}^{n-r}$$

$$\equiv \sum_{r=0}^{n} \sum_{s=0}^{n-r} a_r \binom{n-r}{s} x^{n-s}(1-x)^s$$

$$\equiv \sum_{s=0}^{n} \sum_{r=0}^{n-s} a_r \binom{n-r}{s} x^{n-s}(1-x)^s$$

$$\equiv \sum_{k=0}^{n} \left\{ \sum_{r=0}^{k} a_r \binom{n-r}{n-k} \Big/ \binom{n}{k} \right\} \binom{n}{k} x^k(1-x)^{n-k}$$

$$\equiv \sum_{k=0}^{n} \left\{ \sum_{r=0}^{k} a_r \binom{k}{r} \Big/ \binom{n}{r} \right\} \binom{n}{k} x^k(1-x)^{n-k}.$$

This completes the proof of the theorem.

Next, we seek a more efficient method than the direct use of formula (2) for computing $b_k (k=0, 1, \ldots, n)$. To this end, we note (footnote 3, p. 12) that, if $P(x) \equiv \sum_{r=0}^{n} b_r \binom{n}{r} x^r(1-x)^{n-r} (n \geq 1)$, then, as a simple calculation shows,

$$P'(x) \equiv n \sum_{r=0}^{n-1} \{\Delta b_r\} \binom{n-1}{r} x^r(1-x)^{n-1-r}.$$

Repeating this process, we obtain

$$P^{(k)}(x) \equiv n(n-1)\ldots(n-k+1) \sum_{r=0}^{n-k} \{\Delta^k b_r\} \binom{n-k}{r} x^r(1-x)^{n-k-r} (k=1, 2, \ldots, n). \quad (4)$$

Since $a_k = P^{(k)}(0)/k!$ $(k=1, 2, \ldots, n)$ if $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n$, (4) yields

$$a_k = \binom{n}{k} \Delta^k b_0 \qquad (k=1, 2, \ldots, n). \quad (5)$$

Equation (5) suggests a synthetic method for determining the Bernstein form of a polynomial $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n$ (a_r real, $n \geq 1$, $a_n \neq 0$) without using (2) explicitly. One simply writes the numbers $\Delta^k b_0 = a_k \Big/ \binom{n}{k}$ $(k=0, 1, \ldots, n)$ in their usual positions in a difference table (see table 1) and then completes the table to obtain $b_0, b_1, \ldots, b_n$. In view of (4), the completed difference table also yields bounds for $\{P^{(k)}(x) : 0 \leq x \leq 1\}$ $(k=1, 2, \ldots, n)$.

For example, if $P(x) \equiv 1+10x-20x^2+30x^3+15x^4-x^5$, we start with the incomplete difference table (table 2) and complete it to obtain table 3, which yields $b_0=1$, $b_1=3$, $b_2=3$, $b_3=4$, $b_4=12$, and $b_5=35$. Table 3 also yields the bounds 0 and 115 for P', -40 and 300 for P'', 180 and 480 for P''', and 240 and 360 for $P^{(4)}$ on $[0, 1]$.

TABLE 1

b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	b_n
Δb_0	Δb_1		$\cdots$	Δb_{n-2}	Δb_{n-1}	
$\Delta^2 b_0$			$\cdots$	$\Delta^2 b_{n-2}$		
$\Delta^n b_0$						

TABLE 2

1	*	*	*	*	*
2	*	*	*	*	
-2	*	*	*		
3	*	*			
3	*				
-1					

TABLE 3

1	3	3	4	12	35
2	0	1	8	23	
-2	1	7	15		
3	6	8			
3	2				
-1					

4. Conclusion

We conclude with some miscellaneous remarks.

Formula (2) follows from (5) and the Newton interpolation formula.

One can obviously consider the "Bernstein form" $\sum_{k=0}^{n} b_k \binom{n}{k} x^k(1-x)^{n-k}$ of a polynomial P with complex

[3] Lorentz, G. G., Bernstein Polynomials, University of Toronto Press, Toronto, 1953.

coefficients. It is clear that P maps [0, 1] into the convex hull of the set $\{b_0, b_1, \ldots, b_n\}$.

In view of (5), it is easy to state simple sufficient conditions for a polynomial $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n \equiv \sum_{r=0}^{n} b_r \binom{n}{r} x^r (1-x)^{n-r}$ with real coefficients to have certain properties. For example, if $n \geq 1$, $a_0 > 0$, and $ka_k + (n-k+1)a_{k-1} > 0 (k = 1, 2, \ldots, n)$, then $P(x) > 0$ whenever $0 \leq x \leq 1$. Indeed, $ka_k + (n-k+1)a_{k-1} > 0$ is equivalent to $\Delta^k b_0 + \Delta^{k-1} b_0 = a_k \Big/ \binom{n}{k} + a_{k-1} \Big/ \binom{n}{k-1} > 0$, and, consequently, all the entries in table 1 are positive except possibly $\Delta^k b_0 (k = 1, 2, \ldots, n)$.

Let n be a positive integer. An analog of (1) obviously holds if, for each $k (k = 0, \ldots, n)$, one takes instead of $\binom{n}{k} x^k (1-x)^{n-k}$ a polynomial $q_k(x) \equiv \sum_{r=0}^{n} c_r^{(k)} x^r$ with real coefficients such that (a) $q_k(x) \geq 0$ if $0 \leq x \leq 1$, (b) $q_0, q_1, \ldots, q_n$ are linearly independent, and (c) $\sum_{k=0}^{n} q_k(x) \equiv 1$. It is easy to verify that there are many such sequences $(q_k(x))_{k=0}^{n}$ of polynomials. For example, for each α, $0 < \alpha \leq 1$, the polynomials $P_k(x) \equiv \binom{n}{k} (\alpha x)^k (1 - \alpha x)^{n-k}$ $(k = 0, 1, \ldots, n)$ form such a sequence. Moreover, if $q_0, q_1, \ldots, q_n$ form such a sequence, then so do $Q_0, Q_1, \ldots, Q_n$ where

$$Q_k(x) \equiv \frac{1}{x} \int_0^x q_k(t)\,dt \qquad (k = 0, 1, \ldots, n).$$

(Paper 70B1–168)

796-485 O-66—6

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

Remarks on Measurable Sets and Functions

Roy O. Davies [1]

(October 19, 1965)

A. J. Goldman (On measurable sets and functions, J. Res. NBS **69B** (Math. and Math. Phys.) Nos. 1 and 2, 99–100 (1965)) conjectured that the Borel sets are characterized by their property of having measurable inverse images under all Lebesgue measurable functions: here it is pointed out that the existence of analytic non-Borel sets refutes this and a related conjecture. Also an error in Goldman's Theorem 2 is corrected.

Key Words: Measure, integration, real function.

We deal exclusively with subsets of the real line R, and with real-valued functions having R as domain. Let (BS) and (BF) denote the respective families of Borel sets and Borel-measurable functions, while (LS) and (LF) denote the respective families of Lebesgue-measurable sets and functions. Then $f\epsilon(LF)$ if and only if

$$f^{-1}(B)\epsilon(LS) \qquad \text{for all } B\epsilon(BS). \tag{1}$$

Recently Goldman [2] asked whether (1) characterized (BS), in the sense of the following

CONJECTURE: *If* S *is not in* (BS), *then there is an* fϵ(LF) *such that* f^{-1}(S) *is not in* (LS).

We can *disprove* this conjecture as follows. Let

$$n=\{n(1), n(2), \ldots\}$$

be generic notation for an infinite sequence of positive integers. If $\mathfrak{N}$ is a family of sets, then any set

$$\bigcup_{n}\bigcap_{r=1}^{\infty} F(n(1), \ldots, n(r)),$$

where each $F(n(1), \ldots, n(r))\epsilon\mathfrak{N}$, is said to be "obtained from $\mathfrak{N}$ by operation $(\mathscr{A})$". If $\mathscr{A}(\mathfrak{N})$ consists of all sets obtainable from $\mathfrak{N}$ by operation $(\mathscr{A})$, then for any function f,

$$f^{-1}(\mathscr{A}(\mathfrak{N}))=\mathscr{A}(f^{-1}(\mathfrak{N})). \tag{2}$$

When $\mathfrak{N}=(BS)$, $\mathscr{A}(\mathfrak{N})$ is called the class of *analytic sets*, and it is known [3] that

$$(BS)\subset\mathscr{A}(BS) \text{ but } (BS)\neq\mathscr{A}(BS). \tag{3}$$

For any $f\epsilon(LF)$, it follows from (1) and (2) that

$$f^{-1}(\mathscr{A}(BS))\subset\mathscr{A}(LS). \tag{4}$$

It is also known [4] that (LS) is closed under operation $(\mathscr{A})$, so that (4) implies

$$f^{-1}(\mathscr{A}(BS))\subset(LS) \qquad \text{for all } f\epsilon(LF). \tag{5}$$

Considering $S\epsilon\mathscr{A}(BS)-(BS)$, as permitted by (3), we are led *via* (5) to a contradiction of the conjecture.

Denote functional composition by an asterisk ($(f*g)(x)=f(g(x))$), and let (LCF) be the class of functions f such that

$$g\epsilon(LF) \text{ implies } f*g\epsilon(LF).$$

Goldman (Theorem 4, *op cit*) also showed that we should have

$$(BF)=(LCF) \tag{6}$$

if the Conjecture were true. That (6) fails together with the Conjecture can be proved by choosing as f the characteristic function of some $S\epsilon\mathscr{A}(BS)-(BS)$; clearly f is not in (BF), but for any $B\epsilon(BS)$ we have $f^{-1}(B)$ a member of $\mathscr{A}(BS)$, namely R or ϕ or S or $R-S$.

[1] Department of Mathematics, The University, Leicester, United Kingdom.
[2] A. J. Goldman, On measurable sets and functions, J. Res. NBS **69B** (Math. and Math. Phys.) Nos. 1 and 2, 99–100 (1965).
[3] K. Kuratowski, Topologie **I**, 2d ed. (Warsaw, 1948), p. 391.
[4] K. Kuratowski, *op. cit.*, p. 64.

so that for any $g \in (LF)$ it follows from (5) that

$$(f*g)^{-1}(B) = g^{-1}(f^{-1}(B)) \in (LS),$$

proving $f*g \in (LF)$ and hence $f \in (LCF)$.

Thus the problem of finding a satisfactory characterization of (LCF) remains open. If (QS) is the class of sets Q such that

$$g^{-1}(Q) \in (LS) \qquad \text{for all } g \in (LF),$$

then $f \in (LCF)$ if and only if

$$f^{-1}(B) \in (QS) \qquad \text{for all } B \in (BS).$$

Hence characterizing (LCF) is closely related to characterizing (QS).

Finally, Goldman's Theorem 2 (*op cit*) should be amended to read as follows:

THEOREM: *For any* $B \in (BS)$ *and* $L \in (LS)$, *with sole exceptions* $(B = \phi, L \neq \phi)$ *and* $(B = R, L \neq R)$, *there is an* $f \in (LF)$ *such that* $L = f^{-1}(B)$.

PROOF: If $B = \phi$ and $L = \phi$, or $B = R$ and $L = R$, then any $f \in (LF)$ will do. If $B = \phi$ and $L \neq \phi$, or $B = R$ and $L \neq R$, then no f will do. Finally, if $B \neq \phi$ and $B \neq R$, then we can define f on L so that $f(L) \subset B$, and on $R - L$ so that $f(R - L) \subset R - B$.

(Paper 70B1–169)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

The Form Factor of the Fermi Model Spatial Distribution

L. C. Maximon and R. A. Schrack

(October 19, 1965)

A useful analytic expression for the form factor $F(q)=\int\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}$ of the Fermi distribution $\rho(r)=\rho_0[1+e^{(r-c)/a}]^{-1}$ is derived. This expression consists of a simple term with elementary functions plus a rapidly convergent infinite series with terms of alternating sign. Tables of the form factor as a function of q for several values of the parameters c and a, as well as the numerical values of the normalization constant ρ_0 and the rms radius corresponding to these values of the parameters, are also given.

Key Words: Analytic, fermi distribution, form factor, momentum transfer, normalization constant, root-mean-square radius.

In the calculation of a cross section in the Born approximation the spatial distribution of the interaction may be represented by the inclusion of a form factor which is its Fourier transform. The reaction amplitude for a distributed interaction is then the form factor F times the reaction amplitude for a point interaction. The form factor squared thus enters expressions for the cross section. The concept of the form factor has been utilized in atomic physics for some time and now is used widely in nuclear and particle physics to characterize the effects of the spatial distribution of the interaction.

The form factor F is generally given as a function of the momentum q transferred by the incident particle (represented by a plane wave) to the interacting particle. If the spherically symmetric spatial distribution of the interacting particle is given by $\rho(r)$, then the form factor associated with it is given by the Fourier transform of $\rho(r)$:

$$F(q)=\int\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}=\frac{4\pi}{q}\int_0^\infty\rho(r)r\sin qr\,dr. \tag{1}$$

The density is normalized so that

$$\int\rho(r)d^3\mathbf{r}=1., \tag{2}$$

from which

$$F(0)=1. \tag{3}$$

There are several collections of form factors for various models of the spatial distribution (for example, density distributions in nuclei) in the literature [1, 2, 3, 4].

One of the most widely used models is not included in most collections of form factors. It is defined by the spatial distribution:

$$\rho(r)=\rho_0/(1+\exp(r-c)/a) \tag{4}$$

where the normalization constant ρ_0 is chosen so that (2) is satisfied. The parameters a and c determine the shape of the spatial distribution.

This model has been named variously the Fermi smoothed uniform model, [1] the Fermi 2-parameter model, [4] the smoothed uniform model, [5] the Saxon-Woods distribution, [6] or simply the Fermi distribution [7].

An analytic expression for the form factor of the Fermi distribution has been presented but it is incomplete [8]. In this paper the complete expressions for this form factor, mean square radius of the density distribution and normalization constant will be presented together with tabulated evaluations. The effect of simplifying assumptions is examined.

The mean square radius of the distribution is given by

$$\langle r^2\rangle = \int r^2\rho(r)d^3\mathbf{r} \tag{5}$$

and may be obtained directly from our analytic expression for $F(q)$: Expanding (1) in a power series, we have, for small q,

$$F(q) = 1 - 1/6\langle r^2\rangle q^2 + \ldots \tag{6}$$

so that

$$\langle r^2\rangle = -6\left.\frac{\partial F}{\partial(q^2)}\right|_{q=0} \tag{7}$$

Proceeding now to the derivation of $F(q)$, we have, substituting (4) in (1),

$$F(q) = \frac{4\pi\rho_0}{q}\int_0^\infty \frac{r\sin qr\,dr}{1+e^{(r-c)/a}} = -\frac{4\pi\rho_0}{q}\cdot \mathrm{Re}\,\frac{\partial}{\partial q}\int_0^\infty \frac{e^{iqr}\,dr}{1+e^{(r-c)/a}}. \tag{8}$$

Making the change of variables

$$x = e^{-r/a},\ b = e^{-c/a},\ \beta = qa \tag{9}$$

we then obtain

$$F(q) = -\frac{4\pi\rho_0 a^3}{b\beta}\,\mathrm{Re}\,\frac{\partial}{\partial\beta}\int_0^1 x^{-i\beta}(1+b^{-1}x)^{-1}dx. \tag{10}$$

The integral appearing here is one form of the hypergeometric function: [9]

$${}_2F_1(a,\ b;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{11}$$

Thus

$$\int_0^1 x^{-i\beta}(1+b^{-1}x)^{-1}dx = (1-i\beta)^{-1}\,{}_2F_1(1,\ 1-i\beta;\ 2-i\beta;\ -b^{-1}). \tag{12}$$

Transforming the hypergeometric function in (12) [9]

$${}_2F_1(a,\ b;\ c;\ z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,(-z)^{-a}\,{}_2F_1(a,\ 1-c+a;\ 1-b+a;\ z^{-1})$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,(-z)^{-b}\,{}_2F_1(b,\ 1-c+b;\ 1-a+b;\ z^{-1}) \tag{13}$$

we then have

$$F(q) = -\frac{4\pi\rho_0 a^3}{\beta} \operatorname{Re} \frac{\partial}{\partial\beta} \left\{ -\frac{1}{i\beta} \; {}_2F_1(1,\, i\beta;\, 1+i\beta;\, -b) + \frac{\pi b^{-i\beta}}{i \sinh \pi\beta} \right\} \tag{14}$$

where we have made use of the fact that if either of its first two parameters is zero, the hypergeometric function is unity, and also that [9]

$$\Gamma(1-i\beta)\Gamma(i\beta) = \frac{\pi}{i \sinh \pi\beta}. \tag{15}$$

Using the power series expansion for the hypergeometric function

$${}_2F_1(a,\, b;\, c;\, z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \tag{16}$$

we have, for the first term in (14)

$$-\frac{1}{i\beta} \; {}_2F_1(1,\, i\beta;\, 1+i\beta;\, -b) = -\frac{1}{i\beta} + \frac{b}{1+i\beta} - \frac{b^2}{2+i\beta} + \ldots \tag{17}$$

from which

$$\operatorname{Re} \frac{\partial}{\partial\beta} \left\{ -\frac{1}{i\beta} \; {}_2F_1(1,\, i\beta;\, 1+i\beta;\, -b) \right\} = -2\beta \left[\frac{b}{(1+\beta^2)^2} - \frac{2b^2}{(2^2+\beta^2)^2} + \frac{3b^3}{(3^2+\beta^2)^2} - \ldots \right]. \tag{18}$$

The evaluation of the second term in (14) is straightforward:

$$\operatorname{Re} \frac{\partial}{\partial\beta} \left\{ \frac{\pi b^{-i\beta}}{i \sinh \pi\beta} \right\} = \frac{-\pi}{\beta \sinh^2 \pi\beta} \left[\pi\beta \cosh(\pi\beta) \sin qc - qc \sinh(\pi\beta) \cos qc \right]. \tag{19}$$

Thus, finally, substituting (18) and (19) in (14) we have, in terms of the original parameters,

$$F(q) = \frac{4\pi^2\rho_0 a^3}{(qa)^2 \sinh^2(\pi qa)} \left[\pi qa \cosh(\pi qa) \sin(qc) - qc \cos(qc) \sinh(\pi qa) \right] + 8\pi\rho_0 a^3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n e^{-\frac{nc}{a}}}{[n^2 + (qa)^2]^2}. \tag{20}$$

This expression, without the infinite series, has been given previously by Blankenbecler [8]. To determine ρ_0, we take the limit of (20) as $q \to 0$. From (3), the left-hand side of (20) is unity, so that

$$\rho_0 = \left\{ \frac{4\pi c}{3} \left[(\pi a)^2 + c^2 \right] + 8\pi a^3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-\frac{nc}{a}}}{n^3} \right\}^{-1}. \tag{21}$$

For $\langle r^2 \rangle$, we have, substituting (20) in (7), and carrying out the differentiation,

$$\langle r^2 \rangle = \rho_0 \left\{ \frac{4\pi c}{3} \left[(\pi a)^2 + c^2 \right] \left[\frac{7}{5} (\pi a)^2 + \frac{3}{5} c^2 \right] + 96\pi a^3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-\frac{nc}{a}}}{n^5} \right\}. \tag{22}$$

The expressions (21) and (22) have been given previously by Elton [10].

TABLE 1

c	a	ρ_0	ρ_0'	$\langle r^2\rangle^{1/2}$	$\langle r^2\rangle'^{1/2}$
2.	0.	$2.984*10^{-2}$	$2.984*10^{-2}$	1.5492	1.5492
2.	0.2	$2.716*10^{-2}$	$2.716*10^{-2}$	1.7183	1.7183
2.	.4	$2.139*10^{-2}$	$2.140*10^{-2}$	2.1471	2.1473
2.	.6	$1.576*10^{-2}$	$1.580*10^{-2}$	2.7139	2.7156
2.	.8	$1.143*10^{-2}$	$1.157*10^{-2}$	3.3468	3.3531
2.	1.0	$8.365*10^{-3}$	$8.606*10^{-3}$	4.0129	4.0271
2.	1.2	$6.227*10^{-3}$	$6.554*10^{-3}$	4.6969	4.7220
2.	1.4	$4.725*10^{-3}$	$5.113*10^{-3}$	5.3914	5.4297
2.	1.6	$3.654*10^{-3}$	$4.079*10^{-3}$	6.0925	6.1459
2.	1.8	$2.875*10^{-3}$	$3.318*10^{-3}$	6.7979	6.8679
2.	2.	$2.298*10^{-3}$	$2.745*10^{-3}$	7.5063	7.5941

Calculations of $F(q)$, the normalization constant ρ_0, radius $\langle r^2\rangle$ are given respectively in expressions (20), (21), and (22). It is interesting to see under what conditions the summation terms appearing in these expressions can be dropped without serious effect on the accuracy of the calculation. In table 1 the primed values are obtained by dropping the sum terms, resulting in the following expressions:

$$\langle r^2\rangle' = \frac{1}{5}\left[7(\pi a)^2 + 3c^2\right] \tag{23}$$

$$\rho_0' = \left\{\frac{4\pi c}{3}\left[(\pi a)^2 + c^2\right]\right\}^{-1}. \tag{24}$$

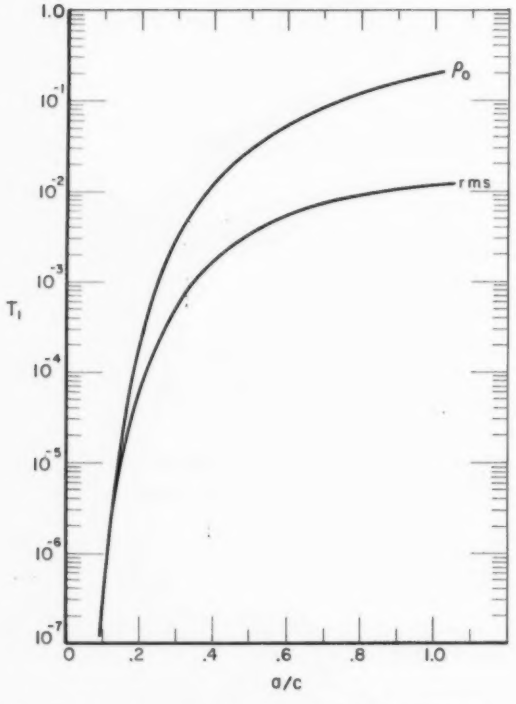

FIGURE 1. *The ordinate is the fraction* T_1 *that the sum terms contribute in the calculation of the normalization constant* ρ_0 *and the rms radius as a function of the ratio* a/c *of the Fermi model parameters.*

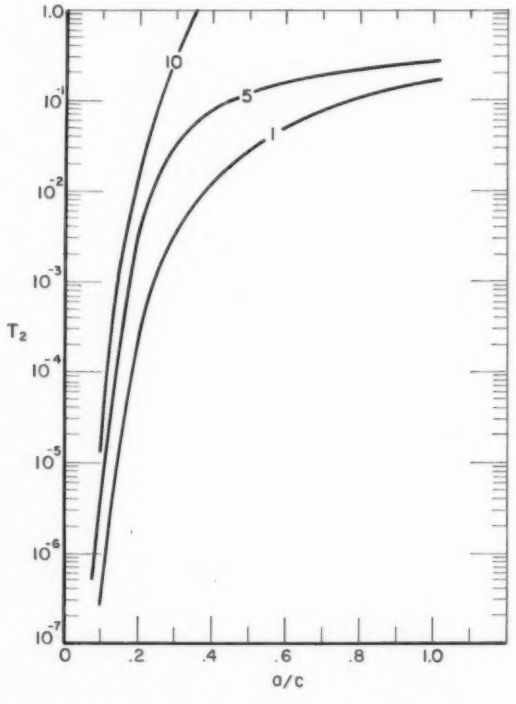

FIGURE 2. *The ordinate is the fraction* T_2 *that the sum terms contribute in the calculation of the form factor as a function of the ratio* a/c *of the Fermi model parameters.*
The curves are labeled with the value of $q\ \langle r^2\rangle^{1/2}$ to which they apply.

TABLE 2

$c = 3.00$ $a = 0.30$
$\rho_0 = 8.047667E-03$ rms = 2.577507E 00

	r/c	r	ρ/ρ₀	ρ
1	0.	0.	10.000E−01	8.047E−03
2	.05	.15	9.999E−01	8.047E−03
3	.10	.30	9.999E−01	8.047E−03
4	.15	.45	9.998E−01	8.046E−03
5	.20	.60	9.997E−01	8.045E−03
6	.25	.75	9.994E−01	8.043E−03
7	.30	.90	9.991E−01	8.040E−03
8	.35	1.05	9.985E−01	8.036E−03
9	.40	1.20	9.975E−01	8.028E−03
10	.45	1.35	9.959E−01	8.015E−03
11	.50	1.50	9.933E−01	7.994E−03
12	.55	1.65	9.890E−01	7.959E−03
13	.60	1.80	9.820E−01	7.903E−03
14	.65	1.95	9.707E−01	7.812E−03
15	.70	2.10	9.526E−01	7.666E−03
16	.75	2.25	9.241E−01	7.437E−03
17	.80	2.40	8.808E−01	7.088E−03
18	.85	2.55	8.176E−01	6.580E−03
19	.90	2.70	7.311E−01	5.883E−03
20	.95	2.85	6.225E−01	5.009E−03
21	1.00	3.00	5.000E−01	4.024E−03
22	1.05	3.15	3.775E−01	3.038E−03
23	1.10	3.30	2.689E−01	2.164E−03
24	1.15	3.45	1.824E−01	1.468E−03
25	1.20	3.60	1.192E−01	9.593E−04
26	1.25	3.75	7.586E−02	6.105E−04
27	1.30	3.90	4.743E−02	3.817E−04
28	1.35	4.05	2.931E−02	2.359E−04
29	1.40	4.20	1.799E−02	1.447E−04
30	1.45	4.35	1.099E−02	8.842E−05
31	1.50	4.50	6.693E−03	5.386E−05
32	1.55	4.65	4.070E−03	3.276E−05
33	1.60	4.80	2.473E−03	1.990E−05
34	1.65	4.95	1.501E−03	1.208E−05
35	1.70	5.10	9.111E−04	7.332E−06
36	1.75	5.25	5.528E−04	4.449E−06
37	1.80	5.40	3.354E−04	2.699E−06
38	1.85	5.55	2.034E−04	1.637E−06
39	1.90	5.70	1.234E−04	9.930E−07
40	1.95	5.85	7.485E−05	6.023E−07
41	2.00	6.00	4.540E−05	3.653E−07
42	2.05	6.15	2.754E−05	2.216E−07
43	2.10	6.30	1.670E−05	1.344E−07
44	2.15	6.45	1.013E−05	8.152E−08
45	2.20	6.60	6.144E−06	4.945E−08
46	2.25	6.75	3.727E−06	2.999E−08
47	2.30	6.90	2.260E−06	1.819E−08
48	2.35	7.05	1.371E−06	1.103E−08
49	2.40	7.20	8.315E−07	6.692E−09
50	2.45	7.35	5.043E−07	4.059E−09
51	2.50	7.50	3.059E−07	2.462E−09

TABLE 3

$c = 3.00$ $a = 0.60$
$\rho_0 = 6.337823E-03$ rms = 3.220687E 00

	r/c	r	ρ/ρ₀	ρ
1	0.	0.	9.933E−01	6.295E−03
2	.05	.15	9.914E−01	6.283E−03
3	.10	.30	9.890E−01	6.268E−03
4	.15	.45	9.859E−01	6.249E−03
5	.20	.60	9.820E−01	6.224E−03
6	.25	.75	9.770E−01	6.192E−03
7	.30	.90	9.707E−01	6.152E−03
8	.35	1.05	9.627E−01	6.101E−03
9	.40	1.20	9.526E−01	6.037E−03
10	.45	1.35	9.399E−01	5.957E−03
11	.50	1.50	9.241E−01	5.857E−03
12	.55	1.65	9.047E−01	5.734E−03
13	.60	1.80	8.808E−01	5.582E−03
14	.65	1.95	8.520E−01	5.400E−03
15	.70	2.10	8.176E−01	5.182E−03
16	.75	2.25	7.773E−01	4.926E−03
17	.80	2.40	7.311E−01	4.633E−03
18	.85	2.55	6.792E−01	4.305E−03
19	.90	2.70	6.225E−01	3.945E−03
20	.95	2.85	5.622E−01	3.563E−03
21	1.00	3.00	5.000E−01	3.169E−03
22	1.05	3.15	4.378E−01	2.775E−03
23	1.10	3.30	3.775E−01	2.393E−03
24	1.15	3.45	3.208E−01	2.033E−03
25	1.20	3.60	2.689E−01	1.705E−03
26	1.25	3.75	2.227E−01	1.411E−03
27	1.30	3.90	1.824E−01	1.156E−03
28	1.35	4.05	1.480E−01	9.383E−04
29	1.40	4.20	1.192E−01	7.555E−04
30	1.45	4.35	9.535E−02	6.043E−04
31	1.50	4.50	7.586E−02	4.808E−04
32	1.55	4.65	6.009E−02	3.808E−04
33	1.60	4.80	4.743E−02	3.006E−04
34	1.65	4.95	3.733E−02	2.366E−04
35	1.70	5.10	2.931E−02	1.858E−04
36	1.75	5.25	2.298E−02	1.456E−04
37	1.80	5.40	1.799E−02	1.140E−04
38	1.85	5.55	1.406E−02	8.913E−05
39	1.90	5.70	1.099E−02	6.963E−05
40	1.95	5.85	8.577E−03	5.436E−05
41	2.00	6.00	6.693E−03	4.242E−05
42	2.05	6.15	5.220E−03	3.308E−05
43	2.10	6.30	4.070E−03	2.580E−05
44	2.15	6.45	3.173E−03	2.011E−05
45	2.20	6.60	2.473E−03	1.567E−05
46	2.25	6.75	1.927E−03	1.221E−05
47	2.30	6.90	1.501E−03	9.514E−06
48	2.35	7.05	1.170E−03	7.412E−06
49	2.40	7.20	9.111E−04	5.774E−06
50	2.45	7.35	7.097E−04	4.498E−06
51	2.50	7.50	5.528E−04	3.503E−06

Figure 1 shows the fraction of the total that the sum terms contribute. In most cases (where $a/c < 0.3$) the short expressions, in which the sum terms are dropped, can be used with adequate accuracy. Figure 2 shows the effect of the sum terms on the form factor calculation. For values of $q \langle r^2 \rangle^{1/2} < 1$ the sum terms can be dropped when $a/c < 0.3$, but for large values of $q \langle r^2 \rangle^{1/2}$ the sum terms become important for all a/c. Tables 2 to 6 give values of the density distribution and tables 7 to 11 values of the form factor for various values of c and a. It should be noted that the model can be scaled. The important parameters in the description of a model are c and a/c. For example, if:

$$c_2 = mc_1 \tag{25}$$

and

$$\frac{a_2}{c_2} = \frac{a_1}{c_1} \tag{26}$$

then

$$\langle r_2^2 \rangle = m^2 \langle r_1^2 \rangle, \tag{27}$$

$$\rho_{0_2} = m^{-3} \rho_{0_1}, \tag{28}$$

TABLE 4

	$c = 3.00$ $\rho_0 = 4.668352\text{E}-03$		$a = 0.90$ $,\text{rms} = 4.070780\text{E}\ 00$	
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	9.656E−01	4.508E−03
2	.05	.15	9.596E−01	4.480E−03
3	.10	.30	9.526E−01	4.447E−03
4	.15	.45	9.445E−01	4.409E−03
5	.20	.60	9.350E−01	4.365E−03
6	.25	.75	9.241E−01	4.314E−03
7	.30	.90	9.116E−01	4.256E−03
8	.35	1.05	8.972E−01	4.189E−03
9	.40	1.20	8.808E−01	4.112E−03
10	.45	1.35	8.622E−01	4.025E−03
11	.50	1.50	8.411E−01	3.927E−03
12	.55	1.65	8.176E−01	3.817E−03
13	.60	1.80	7.914E−01	3.694E−03
14	.65	1.95	7.625E−01	3.560E−03
15	.70	2.10	7.311E−01	3.413E−03
16	.75	2.25	6.971E−01	3.254E−03
17	.80	2.40	6.608E−01	3.085E−03
18	.85	2.55	6.225E−01	2.906E−03
19	.90	2.70	5.826E−01	2.720E−03
20	.95	2.85	5.416E−01	2.528E−03
21	1.00	3.00	5.000E−01	2.334E−03
22	1.05	3.15	4.584E−01	2.140E−03
23	1.10	3.30	4.174E−01	1.949E−03
24	1.15	3.45	3.775E−01	1.762E−03
25	1.20	3.60	3.392E−01	1.584E−03
26	1.25	3.75	3.029E−01	1.414E−03
27	1.30	3.90	2.689E−01	1.256E−03
28	1.35	4.05	2.375E−01	1.109E−03
29	1.40	4.20	2.086E−01	9.739E−04
30	1.45	4.35	1.824E−01	8.516E−04
31	1.50	4.50	1.589E−01	7.417E−04
32	1.55	4.65	1.378E−01	6.435E−04
33	1.60	4.80	1.192E−01	5.565E−04
34	1.65	4.95	1.028E−01	4.798E−04
35	1.70	5.10	8.840E−02	4.127E−04
36	1.75	5.25	7.586E−02	3.541E−04
37	1.80	5.40	6.497E−02	3.033E−04
38	1.85	5.55	5.555E−02	2.593E−04
39	1.90	5.70	4.743E−02	2.214E−04
40	1.95	5.85	4.044E−02	1.888E−04
41	2.00	6.00	3.445E−02	1.608E−04
42	2.05	6.15	2.931E−02	1.368E−04
43	2.10	6.30	2.492E−02	1.164E−04
44	2.15	6.45	2.118E−02	9.887E−05
45	2.20	6.60	1.799E−02	8.397E−05
46	2.25	6.75	1.527E−02	7.127E−05
47	2.30	6.90	1.295E−02	6.047E−05
48	2.35	7.05	1.099E−02	5.129E−05
49	2.40	7.20	9.316E−03	4.349E−05
50	2.45	7.35	7.897E−03	3.687E−05
51	2.50	7.50	6.693E−03	3.124E−05

TABLE 5

	$c = 3.00$ $\rho_0 = 3.387275\text{E}-3$		$a = 1.20$ $\text{rms} = 5.020226\text{E}\ 00$	
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	9.241E−01	3.130E−03
2	.05	.15	9.149E−01	3.099E−03
3	.10	.30	9.047E−01	3.064E−03
4	.15	.45	8.933E−01	3.026E−03
5	.20	.60	8.808E−01	2.984E−03
6	.25	.75	8.670E−01	2.937E−03
7	.30	.90	8.520E−01	2.886E−03
8	.35	1.05	8.355E−01	2.830E−03
9	.40	1.20	8.176E−01	2.769E−03
10	.45	1.35	7.982E−01	2.704E−03
11	.50	1.50	7.773E−01	2.633E−03
12	.55	1.65	7.549E−01	2.557E−03
13	.60	1.80	7.311E−01	2.476E−03
14	.65	1.95	7.058E−01	2.391E−03
15	.70	2.10	6.792E−01	2.301E−03
16	.75	2.25	6.514E−01	2.206E−03
17	.80	2.40	6.225E−01	2.108E−03
18	.85	2.55	5.927E−01	2.008E−03
19	.90	2.70	5.622E−01	1.904E−03
20	.95	2.85	5.312E−01	1.799E−03
21	1.00	3.00	5.000E−01	1.694E−03
22	1.05	3.15	4.688E−01	1.588E−03
23	1.10	3.30	4.378E−01	1.483E−03
24	1.15	3.45	4.073E−01	1.380E−03
25	1.20	3.60	3.775E−01	1.279E−03
26	1.25	3.75	3.486E−01	1.181E−03
27	1.30	3.90	3.208E−01	1.087E−03
28	1.35	4.05	2.942E−01	9.966E−04
29	1.40	4.20	2.689E−01	9.110E−04
30	1.45	4.35	2.451E−01	8.302E−04
31	1.50	4.50	2.227E−01	7.543E−04
32	1.55	4.65	2.018E−01	6.836E−04
33	1.60	4.80	1.824E−01	6.179E−04
34	1.65	4.95	1.645E−01	5.573E−04
35	1.70	5.10	1.480E−01	5.015E−04
36	1.75	5.25	1.330E−01	4.504E−04
37	1.80	5.40	1.192E−01	4.038E−04
38	1.85	5.55	1.067E−01	3.614E−04
39	1.90	5.70	9.535E−02	3.230E−04
40	1.95	5.85	8.510E−02	2.883E−04
41	2.00	6.00	7.586E−02	2.570E−04
42	1.05	6.15	6.755E−02	2.288E−04
43	2.10	6.30	6.009E−02	2.035E−04
44	2.15	6.45	5.340E−02	1.809E−04
45	2.20	6.60	4.743E−02	1.606E−04
46	2.25	6.75	4.209E−02	1.426E−04
47	2.30	6.90	3.733E−02	1.264E−04
48	2.35	7.05	3.309E−02	1.121E−04
49	2.40	7.20	2.931E−02	9.929E−05
50	2.45	7.35	2.596E−02	8.792E−05
51	2.50	7.50	2.298E−02	7.783E−05

and

$$F_2(q) = F_1(mq). \tag{29}$$

The ability to scale the model allows one to use the given tables for any value of the parameter c. The data of table 1 have been reduced to a graph shown in figure 3. This graph shows the relationship of the root-mean-square radius to the model shape (a/c) for the case of $c = 1$. To use the graph for other values of c multiply the abscissa scale by the value of c.

TABLE 6

$c = 3.00$ $a = 1.50$
$\rho_0 = 2.478644\text{E}-03$ rms = 6.019361E 00

	r/c	r	ρ/ρ₀	ρ
1	0.	0.	8.808E−01	2.183E−03
2	.05	.15	8.699E−01	2.156E−03
3	.10	.30	8.581E−01	2.127E−03
4	.15	.45	8.455E−01	2.096E−03
5	.20	.60	8.320E−01	2.062E−03
6	.25	.75	8.176E−01	2.026E−03
7	.30	.90	8.022E−01	1.988E−03
8	.35	1.05	7.858E−01	1.948E−03
9	.40	1.20	7.685E−01	1.905E−03
10	.45	1.35	7.503E−01	1.860E−03
11	.50	1.50	7.311E−01	1.812E−03
12	.55	1.65	7.109E−01	1.762E−03
13	.60	1.80	6.900E−01	1.710E−03
14	.65	1.95	6.682E−01	1.656E−03
15	.70	2.10	6.457E−01	1.600E−03
16	.75	2.25	6.225E−01	1.543E−03
17	.80	2.40	5.987E−01	1.484E−03
18	.85	2.55	5.744E−01	1.424E−03
19	.90	2.70	5.498E−01	1.363E−03
20	.95	2.85	5.250E−01	1.301E−03
21	1.00	3.00	5.000E−01	1.239E−03
22	1.05	3.15	4.750E−01	1.177E−03
23	1.10	3.30	4.502E−01	1.116E−03
24	1.15	3.45	4.256E−01	1.055E−03
25	1.20	3.60	4.013E−01	9.947E−04
26	1.25	3.75	3.775E−01	9.358E−04
27	1.30	3.90	3.543E−01	8.783E−04
28	1.35	4.05	3.318E−01	8.224E−04
29	1.40	4.20	3.100E−01	7.684E−04
30	1.45	4.35	2.891E−01	7.165E−04
31	1.50	4.50	2.689E−01	6.666E−04
32	1.55	4.65	2.497E−01	6.190E−04
33	1.60	4.80	2.315E−01	5.737E−04
34	1.65	4.95	2.142E−01	5.308E−04
35	1.70	5.10	1.978E−01	4.903E−04
36	1.75	5.25	1.824E−01	4.522E−04
37	1.80	5.40	1.680E−01	4.164E−04
38	1.85	5.55	1.545E−01	3.829E−04
39	1.90	5.70	1.419E−01	3.516E−04
40	1.95	5.85	1.301E−01	3.225E−04
41	2.00	6.00	1.192E−01	2.955E−04
42	2.05	6.15	1.091E−01	2.704E−04
43	2.10	6.30	9.975E−02	2.472E−04
44	2.15	6.45	9.112E−02	2.259E−04
45	2.20	6.60	8.317E−02	2.062E−04
46	2.25	6.75	7.586E−02	1.880E−04
47	2.30	6.90	6.914E−02	1.714E−04
48	2.35	7.05	6.297E−02	1.561E−04
49	2.40	7.20	5.732E−02	1.421E−04
50	2.45	7.35	5.215E−02	1.293E−04
51	2.50	7.50	4.743E−02	1.176E−04

TABLE 7

$c = 3.00$ $a = 0.30$
rms radius = 2.578

	q × rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.116	9.851E−01
3	.60	.233	9.414E−01
4	.90	.349	8.721E−01
5	1.20	.466	7.818E−01
6	1.50	.582	6.765E−01
7	1.80	.698	5.630E−01
8	2.10	.815	4.478E−01
9	2.40	.931	3.371E−01
10	2.70	1.048	2.361E−01
11	3.00	1.164	1.485E−01
12	3.30	1.280	7.681E−02
13	3.60	1.397	2.181E−02
14	3.90	1.513	−1.688E−02
15	4.20	1.629	−4.077E−02
16	4.50	1.746	−5.213E−02
17	4.80	1.862	−5.369E−02
18	5.10	1.979	−4.828E−02
19	5.40	2.095	−3.861E−02
20	5.70	2.211	−2.704E−02
21	6.00	2.328	−1.545E−02
22	6.30	2.444	−5.196E−03
23	6.60	2.561	2.887E−03
24	6.90	2.677	8.440E−03
25	7.20	2.793	1.149E−02
26	7.50	2.910	1.234E−02
27	7.80	3.026	1.147E−02
28	8.10	3.143	9.429E−03
29	8.40	3.259	6.764E−03
30	8.70	3.375	3.952E−03
31	9.00	3.492	1.366E−03
32	9.30	3.608	−7.405E−04
33	9.60	3.725	−2.236E−03
34	9.90	3.841	−3.093E−03
35	10.20	3.957	−3.365E−03
36	10.50	4.074	−3.164E−03
37	10.80	4.190	−2.628E−03
38	11.10	4.306	−1.903E−03
39	11.40	4.423	−1.122E−03
40	11.70	4.539	−3.905E−04
41	12.00	4.656	2.143E−04
42	12.30	4.772	6.504E−04
43	12.60	4.888	9.053E−04
44	12.90	5.005	9.911E−04
45	13.20	5.121	9.373E−04
46	13.50	5.238	7.829E−04
47	13.80	5.354	5.699E−04
48	14.10	5.470	3.373E−04
49	14.40	5.587	1.177E−04
50	14.70	5.703	−6.559E−05
51	15.00	5.820	−1.990E−04

TABLE 8

$c = 3.00$ $a = 0.60$
rms radius = 3.221

	$q \times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.093	9.851E−01
3	.60	.186	9.417E−01
4	.90	.279	8.734E−01
5	1.20	.373	7.856E−01
6	1.50	.466	6.850E−01
7	1.80	.559	5.783E−01
8	2.10	.652	4.718E−01
9	2.40	.745	3.711E−01
10	2.70	.838	2.800E−01
11	3.00	.931	2.011E−01
12	3.30	1.025	1.358E−01
13	3.60	1.118	8.383E−02
14	3.90	1.211	4.452E−02
15	4.20	1.304	1.637E−02
16	4.50	1.397	−2.394E−03
17	4.80	1.490	−1.366E−02
18	5.10	1.584	−1.923E−02
19	5.40	1.677	−2.074E−02
20	5.70	1.770	−1.959E−02
21	6.00	1.863	−1.687E−02
22	6.30	1.956	−1.345E−02
23	6.60	2.049	−9.902E−03
24	6.90	2.142	−6.635E−03
25	7.20	2.236	−3.865E−03
26	7.50	2.329	−1.685E−03
27	7.80	2.422	−1.013E−04
28	8.10	2.515	9.428E−04
29	8.40	2.608	1.534E−03
30	8.70	2.701	1.772E−03
31	9.00	2.794	1.757E−03
32	9.30	2.888	1.576E−03
33	9.60	2.981	1.305E−03
34	9.90	3.074	9.991E−04
35	10.20	3.167	7.006E−04
36	10.50	3.260	4.355E−04
37	10.80	3.353	2.178E−04
38	11.10	3.446	5.254E−05
39	11.40	3.540	−6.211E−05
40	11.70	3.633	−1.321E−04
41	12.00	3.726	−1.657E−04
42	12.30	3.819	−1.718E−04
43	12.60	3.912	−1.590E−04
44	12.90	4.005	−1.347E−04
45	13.20	4.099	−1.052E−04
46	13.50	4.192	−7.486E−05
47	13.80	4.285	−4.699E−05
48	14.10	4.378	−2.343E−05
49	14.40	4.471	−5.013E−06
50	14.70	4.564	8.158E−06
51	15.00	4.657	1.653E−05

TABLE 9

$c = 3.00$ $a = 0.90$
rms radius = 4.071

	$q \times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.074	9.851E−01
3	.60	.147	9.419E−01
4	.90	.221	8.744E−01
5	1.20	.295	7.886E−01
6	1.50	.368	6.914E−01
7	1.80	.442	5.895E−01
8	2.10	.516	4.891E−01
9	2.40	.590	3.948E−01
10	2.70	.663	3.101E−01
11	3.00	.737	2.366E−01
12	3.30	.811	1.749E−01
13	3.60	.884	1.249E−01
14	3.90	.958	8.550E−02
15	4.20	1.032	5.544E−02
16	4.50	1.105	3.325E−02
17	4.80	1.179	1.749E−02
18	5.10	1.253	6.787E−03
19	5.40	1.327	−5.941E−05
20	5.70	1.400	−4.065E−03
21	6.00	1.474	−6.060E−03
22	6.30	1.548	−6.700E−03
23	6.60	1.621	−6.483E−03
24	6.90	1.695	−5.777E−03
25	7.20	1.769	−4.839E−03
26	7.50	1.842	−3.845E−03
27	7.80	1.916	−2.902E−03
28	8.10	1.990	−2.070E−03
29	8.40	2.063	−1.378E−03
30	8.70	2.137	−8.289E−04
31	9.00	2.211	−4.148E−04
32	9.30	2.285	−1.183E−04
33	9.60	2.358	8.052E−05
34	9.90	2.432	2.022E−04
35	10.20	2.506	2.658E−04
36	10.50	2.579	2.877E−04
37	10.80	2.653	2.816E−04
38	11.10	2.727	2.583E−04
39	11.40	2.800	2.259E−04
40	11.70	2.874	1.902E−04
41	12.00	2.948	1.551E−04
42	12.30	3.022	1.230E−04
43	12.60	3.095	9.525E−05
44	12.90	3.169	7.223E−05
45	13.20	3.243	5.388E−05
46	13.50	3.316	3.982E−05
47	13.80	3.390	2.945E−05
48	14.10	3.464	2.212E−05
49	14.40	3.537	1.717E−05
50	14.70	3.611	1.403E−05
51	15.00	3.685	1.219E−05

The Model Parameters. The model parameters c and a which appear in the analytic expression (4) for the density distribution are usually identified with geometric concepts. The parameter c is usually thought of as the radius for which the density is one-half the density at zero radius. The parameter a is usually related to the edge thickness t, defined as the measured distance in which the density falls from 0.9 to 0.1 of the density at zero radius. When $a \ll c$ it can be shown that:

$$t = 4a \ln 3 \simeq 4.4a. \tag{30}$$

TABLE 10

$c = 3.00$ $a = 1.20$
rms radius = 5.020

	$q \times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.060	9.851E−01
3	.60	.120	9.421E−01
4	.90	.179	8.751E−01
5	1.20	.239	7.905E−01
6	1.50	.299	6.953E−01
7	1.80	.359	5.964E−01
8	2.10	.418	4.995E−01
9	2.40	.478	4.091E−01
10	2.70	.538	3.279E−01
11	3.00	.598	2.574E−01
12	3.30	.657	1.980E−01
13	3.60	.717	1.492E−01
14	3.90	.777	1.100E−01
15	4.20	.837	7.928E−02
16	4.50	.896	5.565E−02
17	4.80	.956	3.788E−02
18	5.10	1.016	2.482E−02
19	5.40	1.076	1.544E−02
20	5.70	1.135	8.893E−03
21	6.00	1.195	4.473E−03
22	6.30	1.255	1.613E−03
23	6.60	1.315	−1.295E−04
24	6.90	1.374	−1.096E−03
25	7.20	1.434	−1.541E−03
26	7.50	1.494	−1.653E−03
27	7.80	1.554	−1.564E−03
28	8.10	1.613	−1.367E−03
29	8.40	1.673	−1.125E−03
30	8.70	1.733	−8.767E−04
31	9.00	1.793	−6.455E−04
32	9.30	1.853	−4.436E−04
33	9.60	1.912	−2.759E−04
34	9.90	1.972	−1.424E−04
35	10.20	2.032	−4.054E−05
36	10.50	2.092	3.382E−05
37	10.80	2.151	8.526E−05
38	11.10	2.211	1.183E−04
39	11.40	2.271	1.372E−04
40	11.70	2.331	1.455E−04
41	12.00	2.390	1.462E−04
42	12.30	2.450	1.418E−04
43	12.60	2.510	1.342E−04
44	12.90	2.570	1.247E−04
45	13.20	2.629	1.145E−04
46	13.50	2.689	1.042E−04
47	13.80	2.749	9.425E−05
48	14.10	2.809	8.501E−05
49	14.40	2.868	7.660E−05
50	14.70	2.928	6.908E−05
51	15.00	2.988	6.241E−05

TABLE 11

$c = 3.00$ $a = 1.50$
rms radius = 6.019

	$q \times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.050	9.851E−01
3	.60	.100	9.422E−01
4	.90	.150	8.755E−01
5	1.20	.199	7.917E−01
6	1.50	.249	6.978E−01
7	1.80	.299	6.007E−01
8	2.10	.349	5.059E−01
9	2.40	.399	4.178E−01
10	2.70	.449	3.388E−01
11	3.00	.498	2.702E−01
12	3.30	.548	2.121E−01
13	3.60	.598	1.641E−01
14	3.90	.648	1.252E−01
15	4.20	.698	9.414E−02
16	4.50	.748	6.983E−02
17	4.80	.797	5.107E−02
18	5.10	.847	3.679E−02
19	5.40	.897	2.608E−02
20	5.70	.947	1.817E−02
21	6.00	.997	1.241E−02
22	6.30	1.047	8.282E−03
23	6.60	1.096	5.383E−03
24	6.90	1.146	3.387E−03
25	7.20	1.196	2.046E−03
26	7.50	1.246	1.172E−03
27	7.80	1.296	6.259E−04
28	8.10	1.346	3.027E−04
29	8.40	1.395	1.278E−04
30	8.70	1.445	4.782E−05
31	9.00	1.495	2.560E−05
32	9.30	1.545	3.586E−05
33	9.60	1.595	6.194E−05
34	9.90	1.645	9.329E−05
35	10.20	1.695	1.237E−04
36	10.50	1.744	1.496E−04
37	10.80	1.794	1.698E−04
38	11.10	1.844	1.837E−04
39	11.40	1.894	1.919E−04
40	11.70	1.944	1.950E−04
41	12.00	1.994	1.940E−04
42	12.30	2.043	1.898E−04
43	12.60	2.093	1.832E−04
44	12.90	2.143	1.750E−04
45	13.20	2.193	1.657E−04
46	13.50	2.243	1.559E−04
47	13.80	2.293	1.459E−04
48	14.10	2.342	1.360E−04
49	14.40	2.392	1.265E−04
50	14.70	2.442	1.174E−04
51	15.00	2.492	1.088E−04

When $a \simeq c$ the geometrical interpretations of the analytic parameters no longer hold true because the density at zero radius is not ρ_0. For the sake of clarity let c^* and t^* be the actual geometrically measured distances and let

$$R_t = (t - t^*)/t \tag{31}$$

be the fractional change in t and

$$R_c = (c^* - c)/c \tag{32}$$

be the fractional change in c. Figure 4 shows the variation of R_t and R_c as a function of a/c. Note that for $a/c < 0.2$ the fractional change in the parameters is less than 1 percent and thus the usually accepted geometrical interpretations of a and c are valid.

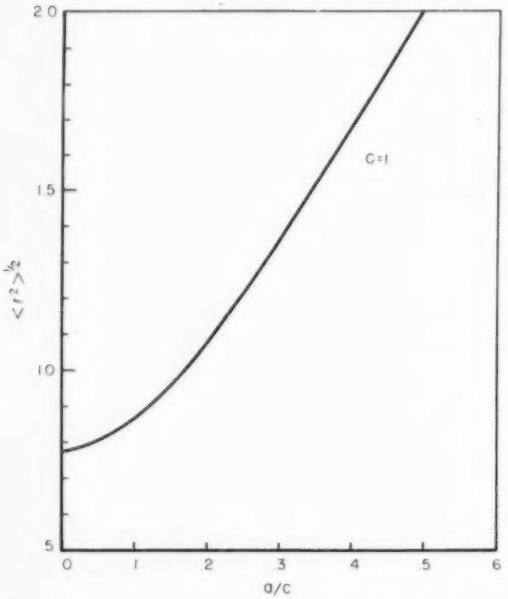

FIGURE 3. *Root-mean-square radius of the spatial distribution for the Fermi distribution with* c = 1. *as a function of* a/c.

To obtain the root-mean-square radius for other values of c simply multiply the result for $c = 1$. by the value of c.

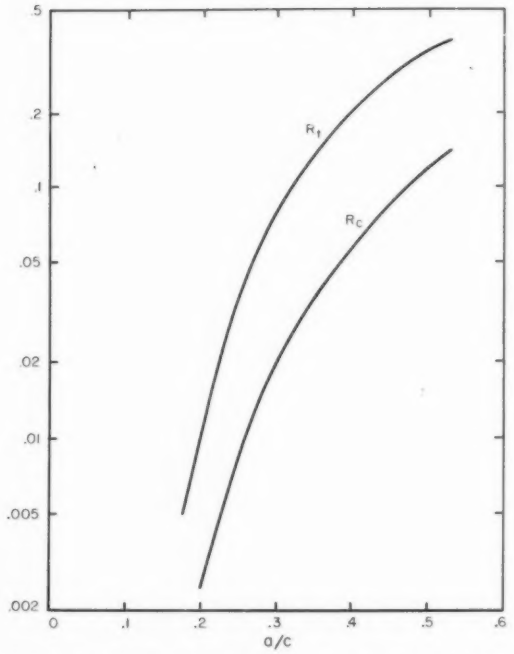

FIGURE 4. *The model parameters* t *and* c *in the analytic representation of the Fermi distribution are normally interpreted in geometrical terms respectively as a measure of the edge thickness and radius at one-half the density at zero radius.*

The curves R_t and R_c indicate respectively the errors in making this correlation between the analytic and geometrical interpretations of t and c.

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[10] It should be noted that if we neglect the infinite sums appearing in both (21) and (22) then we find the result given in ref. 7, on p. 28 eq (2.76) for ρ_0, and on p. 12 eq (2.30) and p. 28 eq (2.77) for $\langle r^2 \rangle$. Note, from p. 107 eq (C.4) that the term $10\pi^2a^2/c$ in eq (2.77) should read $10\pi^2a^2/3c^2$, and that after this correction has been made, the expression written there may be written more simply $R^2 = c^2\left(1 + \frac{7}{3}\frac{\pi^2 a^2}{c^2}\right)$. The complete expressions for the moments of the distribution including the infinite sums, is given on p. 107 eq (C.2). Note, however, that the last terms in this equation should read e^{-mk} rather than e^{m-k}.

(Paper 70B1–170)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

Tables for the Evaluation of the Faxén Approximation to the Solution of the Lamm Equation

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(November 5, 1965)

A table is presented to facilitate the calculation of the Faxén approximation to the concentration and concentration gradient. The table is accurate to within one figure in the last place, and can be used both for no sedimentation dependence on concentration and for the dependence $s = s_0(1 - kc)$.

Key Words: Faxén approximation, Lamm equation, ultracentrifuge, theory of noise, theory of ion exchange columns.

The Faxén approximation to the solution of the Lamm equation has proved to be of considerable utility in the analysis of data from velocity sedimentation experiments with the ultracentrifuge [1].[1] It has recently been shown that the Faxén solution to the nonlinear Lamm equation which arises when the sedimentation coefficient s is related to concentration c by $s = s_0(1 - kc)$ can be written in terms of the Faxén solution for $k = 0$ [2]. Because of this wide range of applicability of the Faxén solution, it is appropriate to present a table to facilitate the calculation of concentration and concentration gradients in that approximation. The best known of earlier, equivalent tables, are those of Opler and Hiester [3]. Their tabulation gives very few values of the relevant function in the range of interest for ultracentrifuge work. A thorough analysis of mathematical properties of the function arising from the Faxén solution has been published by Goldstein [4]. Other applications of the tabulated function are to be found in the theory of noise [5], and in the theory of ion exchange columns [6].

Let us summarize the relevant facts on the Faxén solution. It is assumed that the concentration in the ultracentrifuge is $c(r, t)$, where r is the radius and t the time, with $c(r, 0) = c_0$, a constant. Let r_0 denote radial distance to the meniscus, ω the frequency of revolution, D the diffusion coefficient. The Faxén solution is most conveniently written in terms of the new variables:

$$\theta = c/c_0, \qquad x = (r/r_0)^2, \qquad \tau = 2\omega^2 s_0 t,$$

$$\alpha = kc_0, \qquad \epsilon \doteq 2D/(s_0\omega^2 r_0^2), \qquad z = 2\sqrt{xe^{-\tau}}, \qquad \zeta = 1 - e^{-\tau}. \tag{1}$$

When $k = 0$, or equivalently, when $\alpha = 0$, the Faxén expression for $\theta(z, \zeta)$ is

$$\theta_0(z, \zeta) = \frac{1 - \zeta}{2\epsilon\zeta} \int_z^\infty \sigma e^{-\frac{(\sigma^2 + z^2)}{4\epsilon\zeta}} I_0\left(\frac{z\sigma}{2\epsilon\zeta}\right) d\sigma. \tag{2}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

When $\alpha > 0$ we have

$$\theta_\alpha(z, \zeta) = \left[\frac{1-\zeta}{1-(1+\alpha)\zeta}\right] \frac{\theta_0(z^*, \zeta^*) e^{-\frac{\alpha}{\epsilon}\left[1-\frac{z^2}{4(1-\alpha\zeta)}\right]}}{\left\{1+\frac{1-\alpha\zeta}{1-(1+\alpha)\zeta}\theta_0(z^*, \zeta^*) e^{-\frac{\alpha}{\epsilon}\left[1-\frac{z^2}{4(1-\alpha\zeta)}\right]} - \frac{1}{1-\zeta}\theta_0(z, \zeta)\right\}} \tag{3}$$

where

$$z^* = z(1-\alpha\zeta)^{-1}, \qquad \zeta^* = \zeta(1-\alpha\zeta)^{-1}. \tag{4}$$

Hence $\theta_\alpha(z, \zeta)$ can be represented in terms of $\theta_0(z, \zeta)$.

In the following pages we tabulate values of the functions

$$\psi(y, \eta) = \frac{1}{2\eta}\int_2^\infty \sigma e^{-\frac{(\sigma^2+y^2)}{4\eta}} I_0\left(\frac{\sigma y}{2\eta}\right) d\sigma \tag{5}$$

$$\frac{\partial\psi(y, \eta)}{\partial y} = \frac{1}{4\eta^2}\int_2^\infty \sigma e^{-\frac{(\sigma^2+y^2)}{4\eta}} \left[\sigma I_1\left(\frac{\sigma y}{2\eta}\right) - y I_0\left(\frac{\sigma y}{2\eta}\right)\right] d\sigma \tag{6}$$

accurate to at least four places with a possible error of 1 in the last place. The intervals of tabulation in η are $\eta = 10^{-6}$ (10^{-6}) 10^{-5} (10^{-5}) 10^{-4} (10^{-4}) 10^{-3}. For each value of η we choose from 28 to 80 steps in y so that $\psi(y, \eta)$ lies in the range (0.005, 0.995).

The solutions for concentration and concentration gradients in terms of the tabulated functions are

$$\theta_0(z, \zeta) = (1-\zeta)\psi(z, \epsilon\zeta) \tag{7}$$

$$\frac{\partial\theta_0(z, \zeta)}{\partial(r/r_0)} = 2(1-\zeta)^{3/2}\frac{\partial\psi(z, \epsilon\zeta)}{\partial z} \tag{8}$$

for $\alpha = 0$, and

$$\frac{\partial\theta_\alpha(z, \zeta)}{\partial(r/r_0)} = \frac{1}{(1-\alpha\zeta)F}\frac{\partial\theta_0(z, \zeta)}{\partial(r/r_0)} + \frac{1-\zeta-\theta_0(z, \zeta)}{(1-\alpha\zeta)F^2} \times \left\{\frac{1}{1-(1+\alpha)\zeta}\right.$$
$$\left.\left[(1-\alpha\zeta)\frac{\partial\theta_0(z^*, \zeta^*)}{\partial(r/r_0)} + \frac{\alpha z}{\epsilon}(1-\zeta)^{1/2}\theta_0(z^*, \zeta^*)\right] e^{-\frac{\alpha}{\epsilon}\left[1-\frac{z^2}{4(1-\alpha\zeta)}\right]} - \frac{1}{1-\zeta}\frac{\partial\theta_0(z, \zeta)}{\partial(r/r_0)}\right\} \tag{9}$$

for $\alpha > 0$, where

$$F = 1 + \frac{1-\alpha\zeta}{1-(1+\alpha)\zeta}\theta_0(z^*, \zeta^*) e^{-\frac{\alpha}{\epsilon}\left[1-\frac{z^2}{4(1-\alpha\zeta)}\right]} - \frac{1}{1-\zeta}\theta_0(z, \zeta). \tag{10}$$

The integrals were evaluated using Simpson's rule. The interval of integration h was varied from 2×10^{-5} to 10^{-3} for different values of y and η, so as always to meet the requirements of accuracy. The integration was stopped when the integrand was less than 10^{-25}. As a check on our accuracy, the integration was performed twice; once over the interval $(2, \infty)$ and once over the

interval (0, 2) with the appropriate integrals derived from the formulas

$$\frac{1}{2\eta}\int_2^\infty \sigma e^{-\frac{(\sigma^2+\gamma^2)}{4\eta}} I_0\left(\frac{\sigma\gamma}{2\eta}\right)d\sigma = 1-\frac{1}{2\eta}\int_0^2 \sigma e^{-\frac{(\sigma^2+\gamma^2)}{4\eta}} I_0\left(\frac{\sigma\gamma}{2\eta}\right)d\sigma \tag{11}$$

and

$$\int_2^\infty \sigma e^{-\frac{(\sigma^2+\gamma^2)}{4\eta}}\left[\sigma I_1\left(\frac{\sigma\gamma}{2\eta}\right)-\gamma I_0\left(\frac{\sigma\gamma}{2\eta}\right)\right]d\sigma = -\int_0^2 \sigma e^{-\frac{(\sigma^2+\gamma^2)}{4\eta}}\left[\sigma I_1\left(\frac{\sigma\gamma}{2\eta}\right)-\gamma I_0\left(\frac{\sigma\gamma}{2\eta}\right)\right]d\sigma. \tag{12}$$

Since the argument of the Bessel functions was always greater than 1200, the asymptotic expansions

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1+\frac{1}{1!(8x)}+\frac{3^2}{2!(8x)^2}+\frac{3^2 5^2}{3!(8x)^3}+\ldots\right] \tag{13}$$

$$I_1(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1-\frac{3}{1!8x}-\frac{3.5}{2!(8x)^2}-\frac{3^2 5.7}{3!(8x)^3}-\ldots\right] \tag{14}$$

were used to evaluate these Bessel functions.

All the calculations are accurate to within one figure in the fifth place. Eight place tables are available from the authors, as well as a tabulation for the range $0.001 < \eta \leq 0.01$.

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000001

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9960	0.23414D-02	0.51718D 01	2.0002	0.55637D 00	0.27927D 03
1.9965	0.66707D-02	0.13205D 02	2.0004	0.61149D 00	0.27101D 03
1.9970	0.16962D-01	0.29755D 02	2.0006	0.66444D 00	0.25778D 03
1.9975	0.38579D-01	0.59167D 02	2.0008	0.71432D 00	0.24034D 03
1.9980	0.78701D-01	0.10383D 03	2.0010	0.76036D 00	0.21964D 03
1.9982	0.10161D-00	0.12555D 03	2.0012	0.80203D 00	0.19675D 03
1.9984	0.12902D-00	0.14881D 03	2.0014	0.83899D 00	0.17276D 03
1.9986	0.16119D-00	0.17288D 03	2.0016	0.87112D 00	0.14869D 03
1.9988	0.19817D-00	0.19687D 03	2.0018	0.89852D 00	0.12544D 03
1.9990	0.23986D-00	0.21975D 03	2.0020	0.92140D 00	0.10373D 03
1.9992	0.28592D-00	0.24043D 03	2.0022	0.94014D 00	0.84076D 02
1.9994	0.33581D-00	0.25785D 03	2.0025	0.96148D 00	0.59093D 02
1.9996	0.38878D-00	0.27106D 03	2.0030	0.98307D 00	0.29710D 02
1.9998	0.44391D-00	0.27930D 03	2.0035	0.99334D 00	0.13182D 02
2.0000	0.50014D 00	0.28209D 03	2.0040	0.99766D 00	0.51616D 01

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TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000002

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9950	0.62184D-02	0.87750D 01	2.0002	0.54003D 00	0.19847D 03
1.9955	0.12240D-01	0.15888D 02	2.0004	0.57946D 00	0.19550D 03
1.9960	0.22777D-01	0.27022D 02	2.0006	0.61810D 00	0.19067D 03
1.9965	0.40102D-01	0.43176D 02	2.0008	0.65561D 00	0.18410D 03
1.9970	0.66873D-01	0.64808D 02	2.0010	0.69164D 00	0.17599D 03
1.9972	0.80832D-01	0.74917D 02	2.0012	0.72591D 00	0.16656D 03
1.9974	0.96887D-01	0.85740D 02	2.0014	0.75819D 00	0.15607D 03
1.9976	0.11517D-00	0.97152D 02	2.0016	0.78829D 00	0.14479D 03
1.9978	0.13578D-00	0.10899D 03	2.0018	0.81607D 00	0.13298D 03
1.9980	0.15878D-00	0.12105D 03	2.0020	0.84147D 00	0.12092D 03
1.9982	0.18419D-00	0.13310D 03	2.0022	0.86444D 00	0.10887D 03
1.9984	0.21200D-00	0.14490D 03	2.0024	0.88503D 00	0.97034D 02
1.9986	0.24212D-00	0.15618D 03	2.0026	0.90329D 00	0.85628D 02
1.9988	0.27442D-00	0.16666D 03	2.0028	0.91932D 00	0.74811D 02
1.9990	0.30871D-00	0.17608D 03	2.0030	0.93326D 00	0.64710D 02
1.9992	0.34476D-00	0.18417D 03	2.0032	0.94526D 00	0.55416D 02
1.9994	0.38228D-00	0.19072D 03	2.0035	0.95998D 00	0.43101D 02
1.9996	0.42094D-00	0.19554D 03	2.0040	0.97728D 00	0.26968D 02
1.9998	0.46037D-00	0.19849D 03	2.0045	0.98779D 00	0.15852D 02
2.0000	0.50020D 00	0.19947D 03	2.0050	0.99380D 00	0.87531D 01

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000003

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0002	0.53278D 00	0.16232D 03
1.9935	0.39890D-02	0.48246D 01	2.0004	0.56510D 00	0.16069D 03
1.9940	0.71651D-02	0.81208D 01	2.0006	0.59699D 00	0.15803D 03
1.9945	0.12392D-01	0.13111D 02	2.0008	0.62825D 00	0.15438D 03
1.9950	0.20644D-01	0.20305D 02	2.0010	0.65868D 00	0.14981D 03
1.9955	0.33141D-01	0.30161D 02	2.0012	0.68811D 00	0.14441D 03
1.9960	0.51299D-01	0.42974D 02	2.0014	0.71639D 00	0.13828D 03
1.9965	0.76609D-01	0.58732D 02	2.0016	0.74338D 00	0.13153D 03
1.9968	0.95814D-01	0.69436D 02	2.0018	0.76897D 00	0.12427D 03
1.9970	0.11045D-00	0.76991D 02	2.0020	0.79307D 00	0.11664D 03
1.9972	0.12663D-00	0.84801D 02	2.0022	0.81561D 00	0.10875D 03
1.9974	0.14438D-00	0.92782D 02	2.0024	0.83656D 00	0.10072D 03
1.9976	0.16375D-00	0.10084D 03	2.0026	0.85590D 00	0.92661D 02
1.9978	0.18472D-00	0.10887D 03	2.0028	0.87363D 00	0.84682D 02
1.9980	0.20728D-00	0.11676D 03	2.0030	0.88978D 00	0.76875D 02
1.9982	0.23140D-00	0.12439D 03	2.0032	0.90440D 00	0.69325D 02
1.9984	0.25701D-00	0.13163D 03	2.0034	0.91753D 00	0.62101D 02
1.9986	0.28402D-00	0.13837D 03	2.0037	0.93462D 00	0.51997D 02
1.9988	0.31232D-00	0.14449D 03	2.0040	0.94883D 00	0.42889D 02
1.9990	0.34177D-00	0.14988D 03	2.0045	0.96695D 00	0.30094D 02
1.9992	0.37222D-00	0.15444D 03	2.0050	0.97942D 00	0.20254D 02
1.9994	0.40349D-00	0.15808D 03	2.0055	0.98765D 00	0.13076D 02
1.9996	0.43538D-00	0.16073D 03	2.0060	0.99286D 00	0.80967D 01
1.9998	0.46771D-00	0.16233D 03	2.0065	0.99603D 00	0.48091D 01
2.0000	0.50025D 00	0.16287D 03			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000004

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0002	0.52847D 00	0.14069D 03
			2.0004	0.55651D 00	0.13963D 03
			2.0006	0.58427D 00	0.13789D 03
1.9930	0.66773D-02	0.66084D 01	2.0008	0.61162D 00	0.13549D 03
1.9935	0.10798D-01	0.10075D 02	2.0010	0.63843D 00	0.13247D 03
1.9940	0.16977D-01	0.14888D 02	2.0012	0.66457D 00	0.12887D 03
1.9945	0.25957D-01	0.21324D 02	2.0014	0.68994D 00	0.12474D 03
1.9950	0.38609D-01	0.29602D 02	2.0016	0.71444D 00	0.12014D 03
1.9955	0.55885D-01	0.39829D 02	2.0018	0.73797D 00	0.11514D 03
1.9960	0.78754D-01	0.51940D 02	2.0020	0.76047D 00	0.10979D 03
1.9962	0.89669D-01	0.57257D 02	2.0022	0.78187D 00	0.10417D 03
1.9964	0.10167D-00	0.62803D 02	2.0024	0.80213D 00	0.98346D 02
1.9966	0.11480D-00	0.68542D 02	2.0026	0.82120D 00	0.92383D 02
1.9968	0.12910D-00	0.74433D 02	2.0028	0.83907D 00	0.86349D 02
1.9970	0.14458D-00	0.80427D 02	2.0030	0.85574D 00	0.80306D 02
1.9972	0.16127D-00	0.86470D 02	2.0032	0.87120D 00	0.74314D 02
1.9974	0.17917D-00	0.92504D 02	2.0034	0.88547D 00	0.68425D 02
1.9976	0.19827D-00	0.98465D 02	2.0036	0.89858D 00	0.62689D 02
1.9978	0.21855D-00	0.10429D 03	2.0038	0.91056D 00	0.57148D 02
1.9980	0.23997D-00	0.10990D 03	2.0040	0.92145D 00	0.51836D 02
1.9982	0.26249D-00	0.11524D 03	2.0042	0.93131D 00	0.46784D 02
1.9984	0.28604D-00	0.12024D 03	2.0045	0.94427D 00	0.39740D 02
1.9986	0.31056D-00	0.12483D 03	2.0050	0.96151D 00	0.29528D 02
1.9988	0.33595D-00	0.12895D 03	2.0055	0.97413D 00	0.21266D 02
1.9990	0.36210D-00	0.13253D 03	2.0060	0.98308D 00	0.14844D 02
1.9992	0.38892D-00	0.13554D 03	2.0065	0.98924D 00	0.10043D 02
1.9994	0.41628D-00	0.13793D 03	2.0070	0.99335D 00	0.65854D 01
1.9996	0.44405D-00	0.13966D 03			
1.9998	0.47210D-00	0.14070D 03			
2.0000	0.50028D 00	0.14105D 03			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000005

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9920	0.571890-02	0.515270 01	2.0002	0.525530 00	0.125900 03
1.9925	0.887200-02	0.759050 01	2.0004	0.550640 00	0.125140 03
1.9930	0.134560-01	0.109060 02	2.0006	0.575550 00	0.123890 03
1.9935	0.199550-01	0.152820 02	2.0008	0.600160 00	0.122160 03
1.9940	0.289420-01	0.208850 02	2.0010	0.624380 00	0.119970 03
1.9945	0.410650-01	0.278380 02	2.0012	0.648120 00	0.117360 03
1.9950	0.570140-01	0.361900 02	2.0014	0.671300 00	0.114340 03
1.9955	0.774790-01	0.458860 02	2.0016	0.693840 00	0.110960 03
1.9958	0.921930-01	0.522780 02	2.0018	0.715660 00	0.107240 03
1.9960	0.103090-00	0.567420 02	2.0020	0.736710 00	0.103240 03
1.9962	0.114900-00	0.613420 02	2.0022	0.756940 00	0.989860 02
1.9964	0.127640-00	0.660510 02	2.0024	0.776290 00	0.945310 02
1.9966	0.141320-00	0.708360 02	2.0026	0.794740 00	0.899160 02
1.9968	0.155970-00	0.756650 02	2.0028	0.812250 00	0.851850 02
1.9970	0.171590-00	0.805010 02	2.0030	0.828810 00	0.803810 02
1.9972	0.188170-00	0.853040 02	2.0032	0.844400 00	0.755450 02
1.9974	0.205710-00	0.900330 02	2.0034	0.859030 00	0.707160 02
1.9976	0.224180-00	0.946440 02	2.0036	0.872690 00	0.659320 02
1.9978	0.243550-00	0.990940 02	2.0038	0.885410 00	0.612260 02
1.9980	0.263800-00	0.103340 03	2.0040	0.897190 00	0.566290 02
1.9982	0.284870-00	0.107340 03	2.0042	0.908070 00	0.521680 02
1.9984	0.306720-00	0.111040 03	2.0044	0.918070 00	0.478670 02
1.9986	0.329270-00	0.114420 03	2.0047	0.931500 00	0.417560 02
1.9988	0.352460-00	0.117430 03	2.0050	0.943170 00	0.360990 02
1.9990	0.376210-00	0.120030 03	2.0055	0.959070 00	0.277620 02
1.9992	0.400450-00	0.122210 03	2.0060	0.971160 00	0.208220 02
1.9994	0.425070-00	0.123920 03	2.0065	0.980120 00	0.152320 02
1.9996	0.449980-00	0.125160 03	2.0070	0.986600 00	0.108680 02
1.9998	0.475100-00	0.125910 03	2.0075	0.991170 00	0.756210 01
2.0000	0.500310 00	0.126160 03	2.0080	0.994310 00	0.513220 01

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000006

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0002	0.52336D 00	0.11497D 03
			2.0004	0.54631D 00	0.11439D 03
1.9910	0.46992D-02	0.39496D 01	2.0006	0.56910D 00	0.11343D 03
1.9915	0.70860D-02	0.56864D 01	2.0008	0.59166D 00	0.11211D 03
1.9920	0.10485D-01	0.80181D 01	2.0010	0.61392D 00	0.11044D 03
1.9925	0.15225D-01	0.11373D 02	2.0012	0.63581D 00	0.10843D 03
1.9930	0.21699D-01	0.14976D 02	2.0014	0.65727D 00	0.10610D 03
1.9935	0.30360D-01	0.19837D 02	2.0016	0.67823D 00	0.10347D 03
1.9940	0.41709D-01	0.25735D 02	2.0018	0.69864D 00	0.10058D 03
1.9945	0.56274D-01	0.32698D 02	2.0020	0.71844D 00	0.97436D 02
1.9950	0.74579D-01	0.40689D 02	2.0022	0.73760D 00	0.94080D 02
1.9953	0.87564D-01	0.45930D 02	2.0024	0.75606D 00	0.90537D 02
1.9956	0.10217D-00	0.51460D 02	2.0026	0.77380D 00	0.86838D 02
1.9958	0.11284D-00	0.55280D 02	2.0028	0.79079D 00	0.83013D 02
1.9960	0.12428D-00	0.59187D 02	2.0030	0.80700D 00	0.79092D 02
1.9962	0.13652D-00	0.63158D 02	2.0032	0.82242D 00	0.75106D 02
1.9964	0.14955D-00	0.67172D 02	2.0034	0.83704D 00	0.71083D 02
1.9966	0.16339D-00	0.71204D 02	2.0036	0.85085D 00	0.67052D 02
1.9968	0.17803D-00	0.75226D 02	2.0038	0.86386D 00	0.63039D 02
1.9970	0.19348D-00	0.79211D 02	2.0040	0.87607D 00	0.59068D 02
1.9972	0.20971D-00	0.83129D 02	2.0042	0.88749D 00	0.55164D 02
1.9974	0.22672D-00	0.86951D 02	2.0044	0.89814D 00	0.51347D 02
1.9976	0.24448D-00	0.90646D 02	2.0046	0.90804D 00	0.47634D 02
1.9978	0.26297D-00	0.94184D 02	2.0048	0.91720D 00	0.44043D 02
1.9980	0.28214D-00	0.97533D 02	2.0050	0.92566D 00	0.40587D 02
1.9982	0.30197D-00	0.10067D 03	2.0055	0.94392D 00	0.32608D 02
1.9984	0.32239D-00	0.10355D 03	2.0060	0.95844D 00	0.25658D 02
1.9986	0.34337D-00	0.10617D 03	2.0065	0.96976D 00	0.19773D 02
1.9988	0.36484D-00	0.10849D 03	2.0070	0.97839D 00	0.14924D 02
1.9990	0.38675D-00	0.11049D 03	2.0075	0.98484D 00	0.11031D 02
1.9992	0.40902D-00	0.11216D 03	2.0080	0.98956D 00	0.79860D 01
1.9994	0.43159D-00	0.11347D 03	2.0085	0.99295D 00	0.56622D 01
1.9996	0.45438D-00	0.11441D 03	2.0090	0.99532D 00	0.39318D 01
1.9998	0.47732D-00	0.11498D 03			
2.0000	0.50035D 00	0.11516D 03			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000007

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9905	0.55736D-02	0.42565D 01	2.0002	0.52169D 00	0.10646D 03
1.9910	0.80991D-02	0.59220D 01	2.0004	0.54294D 00	0.10600D 03
1.9915	0.11580D-01	0.80934D 01	2.0006	0.56407D 00	0.10524D 03
1.9920	0.16293D-01	0.10865D 02	2.0008	0.58502D 00	0.10419D 03
1.9925	0.22560D-01	0.14328D 02	2.0010	0.60573D 00	0.10286D 03
1.9930	0.30749D-01	0.18560D 02	2.0012	0.62614D 00	0.10125D 03
1.9935	0.41259D-01	0.23617D 02	2.0014	0.64621D 00	0.99379D 02
1.9940	0.54508D-01	0.29520D 02	2.0016	0.66588D 00	0.97267D 02
1.9945	0.70916D-01	0.36245D 02	2.0018	0.68510D 00	0.94928D 02
1.9950	0.90877D-01	0.43714D 02	2.0020	0.70383D 00	0.92382D 02
1.9952	0.99936D-01	0.46882D 02	2.0022	0.72204D 00	0.89647D 02
1.9954	0.10964D-00	0.50135D 02	2.0024	0.73968D 00	0.86745D 02
1.9956	0.11999D-00	0.53461D 02	2.0026	0.75673D 00	0.83698D 02
1.9958	0.13102D-00	0.56845D 02	2.0028	0.77315D 00	0.80527D 02
1.9960	0.14274D-00	0.60271D 02	2.0030	0.78893D 00	0.77255D 02
1.9962	0.15513D-00	0.63721D 02	2.0032	0.80405D 00	0.73905D 02
1.9964	0.16822D-00	0.67177D 02	2.0034	0.81849D 00	0.70498D 02
1.9966	0.18200D-00	0.70618D 02	2.0036	0.83224D 00	0.67056D 02
1.9968	0.19647D-00	0.74023D 02	2.0038	0.84531D 00	0.63601D 02
1.9970	0.21161D-00	0.77371D 02	2.0040	0.85769D 00	0.60151D 02
1.9972	0.22741D-00	0.80639D 02	2.0042	0.86937D 00	0.56726D 02
1.9974	0.24386D-00	0.83806D 02	2.0044	0.88038D 00	0.53344D 02
1.9976	0.26093D-00	0.86849D 02	2.0046	0.89071D 00	0.50020D 02
1.9978	0.27859D-00	0.89746D 02	2.0048	0.90039D 00	0.46770D 02
1.9980	0.29681D-00	0.92474D 02	2.0050	0.90943D 00	0.43605D 02
1.9982	0.31557D-00	0.95014D 02	2.0052	0.91784D 00	0.40539D 02
1.9984	0.33480D-00	0.97345D 02	2.0055	0.92934D 00	0.36146D 02
1.9986	0.35449D-00	0.99448D 02	2.0060	0.94570D 00	0.29432D 02
1.9988	0.37457D-00	0.10131D 03	2.0065	0.95891D 00	0.23541D 02
1.9990	0.39499D-00	0.10291D 03	2.0070	0.96938D 00	0.18496D 02
1.9992	0.41571D-00	0.10423D 03	2.0075	0.97754D 00	0.14275D 02
1.9994	0.43667D-00	0.10528D 03	2.0080	0.98378D 00	0.10822D 02
1.9996	0.45780D-00	0.10602D 03	2.0085	0.98848D 00	0.80591D 01
1.9998	0.47906D-00	0.10647D 03	2.0090	0.99194D 00	0.58954D 01
2.0000	0.50037D 00	0.10662D 03	2.0095	0.99446D 00	0.42363D 01

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000008

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0002	0.52034D 00	0.99606D 02
			2.0004	0.54022D 00	0.99228D 02
			2.0006	0.56001D 00	0.98605D 02
1.9895	0.43452D-02	0.31894D 01	2.0008	0.57965D 00	0.97741D 02
1.9900	0.62272D-02	0.43930D 01	2.0010	0.59909D 00	0.96643D 02
1.9905	0.87982D-02	0.59571D 01	2.0012	0.61829D 00	0.95318D 02
1.9910	0.12256D-01	0.79528D 01	2.0014	0.63721D 00	0.93777D 02
1.9915	0.16835D-01	0.10452D 02	2.0016	0.65579D 00	0.92031D 02
1.9920	0.22804D-01	0.13525D 02	2.0018	0.67400D 00	0.90091D 02
1.9925	0.30465D-01	0.17229D 02	2.0020	0.69181D 00	0.87972D 02
1.9930	0.40145D-01	0.21607D 02	2.0022	0.70918D 00	0.85689D 02
1.9935	0.52188D-01	0.26678D 02	2.0024	0.72608D 00	0.83256D 02
1.9940	0.66937D-01	0.32428D 02	2.0026	0.74248D 00	0.80691D 02
1.9945	0.84721D-01	0.38806D 02	2.0028	0.75835D 00	0.78009D 02
1.9950	0.10583D-00	0.45719D 02	2.0030	0.77367D 00	0.75228D 02
1.9952	0.11526D-00	0.48605D 02	2.0032	0.78843D 00	0.72365D 02
1.9954	0.12528D-00	0.51543D 02	2.0034	0.80261D 00	0.69437D 02
1.9956	0.13588D-00	0.54523D 02	2.0036	0.81621D 00	0.66461D 02
1.9958	0.14709D-00	0.57531D 02	2.0038	0.82920D 00	0.63454D 02
1.9960	0.15890D-00	0.60553D 02	2.0040	0.84159D 00	0.60432D 02
1.9962	0.17131D-00	0.63575D 02	2.0042	0.85337D 00	0.57410D 02
1.9964	0.18433D-00	0.66581D 02	2.0044	0.86455D 00	0.54403D 02
1.9966	0.19794D-00	0.69555D 02	2.0046	0.87513D 00	0.51425D 02
1.9968	0.21215D-00	0.72481D 02	2.0048	0.88512D 00	0.48488D 02
1.9970	0.22693D-00	0.75341D 02	2.0050	0.89453D 00	0.45605D 02
1.9972	0.24228D-00	0.78118D 02	2.0052	0.90337D 00	0.42786D 02
1.9974	0.25817D-00	0.80795D 02	2.0055	0.91559D 00	0.38700D 02
1.9976	0.27459D-00	0.83356D 02	2.0060	0.93332D 00	0.32331D 02
1.9978	0.29150D-00	0.85783D 02	2.0065	0.94803D 00	0.26591D 02
1.9980	0.30889D-00	0.88060D 02	2.0070	0.96003D 00	0.21532D 02
1.9982	0.32672D-00	0.90172D 02	2.0075	0.96967D 00	0.17164D 02
1.9984	0.34495D-00	0.92104D 02	2.0080	0.97730D 00	0.13471D 02
1.9986	0.36354D-00	0.93843D 02	2.0085	0.98325D 00	0.10408D 02
1.9988	0.38247D-00	0.95375D 02	2.0090	0.98781D 00	0.79170D 01
1.9990	0.40168D-00	0.96691D 02	2.0095	0.99125D 00	0.59289D 01
1.9992	0.42113D-00	0.97780D 02	2.0100	0.99381D 00	0.43711D 01
1.9994	0.44078D-00	0.98634D 02	2.0105	0.99568D 00	0.31727D 01
1.9996	0.46057D-00	0.99248D 02			
1.9998	0.48046D-00	0.99616D 02			
2.0000	0.50040D 00	0.99735D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000009

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0002	0.51922D 00	0.93922D 02
1.9890	0.47757D-02	0.32716D 01	2.0004	0.53798D 00	0.93605D 02
1.9895	0.66840D-02	0.44095D 01	2.0006	0.55665D 00	0.93082D 02
1.9900	0.92374D-02	0.58612D 01	2.0008	0.57520D 00	0.92356D 02
1.9905	0.12607D-01	0.76834D 01	2.0010	0.59358D 00	0.91433D 02
1.9910	0.16992D-01	0.99332D 01	2.0012	0.61176D 00	0.90317D 02
1.9915	0.22620D-01	0.12665D 02	2.0014	0.62969D 00	0.89018D 02
1.9920	0.29745D-01	0.15924D 02	2.0016	0.64735D 00	0.87542D 02
1.9925	0.38639D-01	0.19747D 02	2.0018	0.66470D 00	0.85900D 02
1.9930	0.49589D-01	0.24150D 02	2.0020	0.68170D 00	0.84101D 02
1.9935	0.62884D-01	0.29126D 02	2.0022	0.69833D 00	0.82157D 02
1.9940	0.78805D-01	0.34644D 02	2.0024	0.71456D 00	0.80080D 02
1.9945	0.97609D-01	0.40639D 02	2.0026	0.73035D 00	0.77883D 02
1.9948	0.11036D-00	0.44426D 02	2.0028	0.74570D 00	0.75577D 02
1.9950	0.11951D-00	0.47014D 02	2.0030	0.76058D 00	0.73177D 02
1.9952	0.12917D-00	0.49642D 02	2.0032	0.77497D 00	0.70696D 02
1.9954	0.13937D-00	0.52300D 02	2.0034	0.78885D 00	0.68147D 02
1.9956	0.15009D-00	0.54979D 02	2.0036	0.80222D 00	0.65545D 02
1.9958	0.16136D-00	0.57667D 02	2.0038	0.81507D 00	0.62901D 02
1.9960	0.17316D-00	0.60352D 02	2.0040	0.82738D 00	0.60231D 02
1.9962	0.18550D-00	0.63021D 02	2.0042	0.83916D 00	0.57546D 02
1.9964	0.19837D-00	0.65663D 02	2.0044	0.85040D 00	0.54858D 02
1.9966	0.21176D-00	0.68263D 02	2.0046	0.86110D 00	0.52180D 02
1.9968	0.22567D-00	0.70809D 02	2.0048	0.87127D 00	0.49523D 02
1.9970	0.24008D-00	0.73287D 02	2.0050	0.88091D 00	0.46896D 02
1.9972	0.25498D-00	0.75683D 02	2.0052	0.89003D 00	0.44310D 02
1.9974	0.27035D-00	0.77984D 02	2.0054	0.89864D 00	0.41774D 02
1.9976	0.28616D-00	0.80176D 02	2.0057	0.91062D 00	0.38081D 02
1.9978	0.30241D-00	0.82248D 02	2.0060	0.92151D 00	0.34541D 02
1.9980	0.31905D-00	0.84185D 02	2.0065	0.93738D 00	0.29032D 02
1.9982	0.33607D-00	0.85977D 02	2.0070	0.95063D 00	0.24065D 02
1.9984	0.35344D-00	0.87612D 02	2.0075	0.96154D 00	0.19673D 02
1.9986	0.37111D-00	0.89080D 02	2.0080	0.97040D 00	0.15861D 02
1.9988	0.38906D-00	0.90372D 02	2.0085	0.97749D 00	0.12611D 02
1.9990	0.40724D-00	0.91478D 02	2.0090	0.98310D 00	0.98887D 01
1.9992	0.42563D-00	0.92393D 02	2.0095	0.98746D 00	0.76471D 01
1.9994	0.44419D-00	0.93110D 02	2.0100	0.99082D 00	0.58320D 01
1.9996	0.46286D-00	0.93624D 02	2.0105	0.99336D 00	0.43865D 01
1.9998	0.48162D-00	0.93932D 02	2.0110	0.99525D 00	0.32537D 01
2.0000	0.50042D 00	0.94031D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000010

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9885	0.50797D-02	0.32791D 01	2.0002	0.51828D 00	0.89112D 02
1.9890	0.69749D-02	0.43436D 01	2.0004	0.53608D 00	0.88841D 02
1.9895	0.94690D-02	0.56822D 01	2.0006	0.55381D 00	0.88394D 02
1.9900	0.12710D-01	0.73408D 01	2.0008	0.57142D 00	0.87773D 02
1.9905	0.16871D-01	0.93659D 01	2.0010	0.58890D 00	0.86982D 02
1.9910	0.22145D-01	0.11801D 02	2.0012	0.60621D 00	0.86026D 02
1.9915	0.28747D-01	0.14685D 02	2.0014	0.62330D 00	0.84911D 02
1.9920	0.36909D-01	0.18047D 02	2.0016	0.64016D 00	0.83642D 02
1.9925	0.46876D-01	0.21902D 02	2.0018	0.65675D 00	0.82228D 02
1.9930	0.58894D-01	0.26251D 02	2.0020	0.67304D 00	0.80677D 02
1.9935	0.73206D-01	0.31073D 02	2.0022	0.68901D 00	0.78996D 02
1.9940	0.90038D-01	0.36323D 02	2.0024	0.70463D 00	0.77196D 02
1.9945	0.10959D-00	0.41933D 02	2.0026	0.71988D 00	0.75286D 02
1.9948	0.12269D-00	0.45433D 02	2.0028	0.73474D 00	0.73277D 02
1.9950	0.13202D-00	0.47808D 02	2.0030	0.74919D 00	0.71179D 02
1.9952	0.14182D-00	0.50207D 02	2.0032	0.76321D 00	0.69003D 02
1.9954	0.15210D-00	0.52620D 02	2.0034	0.77678D 00	0.66760D 02
1.9956	0.16286D-00	0.55039D 02	2.0036	0.78991D 00	0.64460D 02
1.9958	0.17411D-00	0.57455D 02	2.0038	0.80257D 00	0.62116D 02
1.9960	0.18585D-00	0.59856D 02	2.0040	0.81475D 00	0.59737D 02
1.9962	0.19806D-00	0.62234D 02	2.0042	0.82646D 00	0.57334D 02
1.9964	0.21074D-00	0.64576D 02	2.0044	0.83768D 00	0.54919D 02
1.9966	0.22388D-00	0.66873D 02	2.0046	0.84843D 00	0.52499D 02
1.9968	0.23748D-00	0.69113D 02	2.0048	0.85868D 00	0.50086D 02
1.9970	0.25152D-00	0.71286D 02	2.0050	0.86846D 00	0.47689D 02
1.9972	0.26599D-00	0.73380D 02	2.0052	0.87776D 00	0.45316D 02
1.9974	0.28087D-00	0.75384D 02	2.0054	0.88659D 00	0.42974D 02
1.9976	0.29614D-00	0.77289D 02	2.0057	0.89896D 00	0.39539D 02
1.9978	0.31178D-00	0.79083D 02	2.0060	0.91032D 00	0.36214D 02
1.9980	0.32776D-00	0.80757D 02	2.0065	0.92710D 00	0.30972D 02
1.9982	0.34407D-00	0.82302D 02	2.0070	0.94137D 00	0.26159D 02
1.9984	0.36068D-00	0.83709D 02	2.0075	0.95334D 00	0.21820D 02
1.9986	0.37755D-00	0.84970D 02	2.0080	0.96327D 00	0.17975D 02
1.9988	0.39465D-00	0.86078D 02	2.0085	0.97140D 00	0.14623D 02
1.9990	0.41197D-00	0.87025D 02	2.0090	0.97797D 00	0.11748D 02
1.9992	0.42945D-00	0.87808D 02	2.0095	0.98322D 00	0.93216D 01
1.9994	0.44708D-00	0.88420D 02	2.0100	0.98736D 00	0.73043D 01
1.9996	0.46481D-00	0.88859D 02	2.0105	0.99059D 00	0.56524D 01
1.9998	0.48261D-00	0.89121D 02	2.0110	0.99307D 00	0.43198D 01
2.0000	0.50045D 00	0.89206D 02	2.0115	0.99495D 00	0.32604D 01

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000020

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.56344D 00	0.62279D 02
1.9825	0.28426D-02	0.13780D 01	2.0020	0.62469D 00	0.59972D 02
1.9850	0.88911D-02	0.38024D 01	2.0030	0.68294D 00	0.56324D 02
1.9875	0.24143D-01	0.89744D 01	2.0040	0.73697D 00	0.51592D 02
1.9900	0.57105D-01	0.18118D 02	2.0050	0.78586D 00	0.46091D 02
1.9910	0.77595D-01	0.22969D 02	2.0060	0.82901D 00	0.40160D 02
1.9920	0.10324D-00	0.28400D 02	2.0070	0.86615D 00	0.34128D 02
1.9930	0.13453D-00	0.34248D 02	2.0080	0.89733D 00	0.28286D 02
1.9940	0.17179D-00	0.40281D 02	2.0090	0.92286D 00	0.22865D 02
1.9950	0.21506D-00	0.46207D 02	2.0100	0.94326D 00	0.18027D 02
1.9960	0.26406D-00	0.51696D 02	2.0110	0.95914D 00	0.13862D 02
1.9970	0.31819D-00	0.56409D 02	2.0125	0.97604D 00	0.89185D 01
1.9980	0.37652D-00	0.60032D 02	2.0150	0.99118D 00	0.37740D 01
1.9990	0.43781D-00	0.62310D 02	2.0175	0.99718D 00	0.13660D 01
2.0000	0.50063D 00	0.63078D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000030

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9800	0.49394D-02	0.18466D 01	2.0010	0.55213D 00	0.51063D 02
1.9825	0.11995D-01	0.40307D 01	2.0020	0.60262D 00	0.49790D 02
1.9850	0.26523D-01	0.79280D 01	2.0030	0.65145D 00	0.47746D 02
1.9875	0.53503D-01	0.14051D 02	2.0040	0.69789D 00	0.45029D 02
1.9900	0.98690D-01	0.22439D 02	2.0050	0.74132D 00	0.41765D 02
1.9910	0.12303D-00	0.26282D 02	2.0060	0.78128D 00	0.38097D 02
1.9920	0.15130D-00	0.30275D 02	2.0070	0.81743D 00	0.34177D 02
1.9930	0.18359D-00	0.34297D 02	2.0080	0.84960D 00	0.30154D 02
1.9940	0.21986D-00	0.38212D 02	2.0090	0.87775D 00	0.26164D 02
1.9950	0.25993D-00	0.41869D 02	2.0100	0.90198D 00	0.22327D 02
1.9960	0.30347D-00	0.45119D 02	2.0110	0.92249D 00	0.18738D 02
1.9970	0.34999D-00	0.47817D 02	2.0125	0.94692D 00	0.13963D 02
1.9980	0.39887D-00	0.49839D 02	2.0150	0.97371D 00	0.78687D 01
1.9990	0.44941D-00	0.51088D 02	2.0175	0.98813D 00	0.39956D 01
2.0000	0.50077D 00	0.51503D 02	2.0200	0.99512D 00	0.18282D 01

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000040

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.54540D 00	0.44314D 02
			2.0020	0.58934D 00	0.43480D 02
			2.0030	0.63218D 00	0.42132D 02
			2.0040	0.67345D 00	0.40318D 02
1.9775	0.59799D-02	0.18953D 01	2.0050	0.71269D 00	0.38103D 02
1.9800	0.12747D-01	0.36797D 01	2.0060	0.74954D 00	0.35563D 02
1.9825	0.25332D-01	0.66070D 01	2.0070	0.78373D 00	0.32779D 02
1.9875	0.81463D-01	0.16850D 02	2.0080	0.81505D 00	0.29838D 02
1.9890	0.10980D-00	0.20995D 02	2.0090	0.84338D 00	0.26824D 02
1.9900	0.13226D-00	0.23934D 02	2.0100	0.86870D 00	0.23815D 02
1.9910	0.15769D-00	0.26945D 02	2.0110	0.89104D 00	0.20880D 02
1.9920	0.18615D-00	0.29958D 02	2.0125	0.91921D 00	0.16745D 02
1.9930	0.21758D-00	0.32894D 02	2.0150	0.95345D 00	0.10890D 02
1.9940	0.25188D-00	0.35670D 02	2.0175	0.97493D 00	0.65494D 01
1.9950	0.28884D-00	0.38199D 02	2.0200	0.98740D 00	0.36431D 01
1.9960	0.32817D-00	0.40399D 02	2.0225	0.99410D 00	0.18741D 01
1.9970	0.36950D-00	0.42195D 02			
1.9980	0.41240D-00	0.43523D 02			
1.9990	0.45638D-00	0.44336D 02			
2.0000	0.50089D 00	0.44603D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000050

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.54082D 00	0.39685D 02
1.9750	0.62539D-02	0.17639D 01	2.0020	0.58024D 00	0.39084D 02
1.9775	0.12304D-01	0.31919D 01	2.0030	0.61886D 00	0.38110D 02
1.9800	0.22886D-01	0.54262D 01	2.0040	0.65634D 00	0.36790D 02
1.9825	0.40276D-01	0.86656D 01	2.0050	0.69234D 00	0.35162D 02
1.9850	0.67133D-01	0.13001D 02	2.0060	0.72658D 00	0.33272D 02
1.9865	0.88911D-01	0.16093D 02	2.0070	0.75881D 00	0.31171D 02
1.9880	0.11556D-00	0.19477D 02	2.0080	0.78887D 00	0.28911D 02
1.9890	0.13621D-00	0.21845D 02	2.0090	0.81660D 00	0.26549D 02
1.9900	0.15926D-00	0.24258D 02	2.0100	0.84195D 00	0.24137D 02
1.9910	0.18473D-00	0.26668D 02	2.0110	0.86488D 00	0.21725D 02
1.9920	0.21258D-00	0.29027D 02	2.0120	0.88541D 00	0.19360D 02
1.9930	0.24275D-00	0.31280D 02	2.0130	0.90363D 00	0.17081D 02
1.9940	0.27509D-00	0.33372D 02	2.0140	0.91962D 00	0.14920D 02
1.9950	0.30942D-00	0.35250D 02	2.0150	0.93351D 00	0.12903D 02
1.9960	0.34550D-00	0.36864D 02	2.0175	0.96016D 00	0.85901D 01
1.9970	0.38304D-00	0.38167D 02	2.0200	0.97738D 00	0.53722D 01
1.9980	0.42172D-00	0.39123D 02	2.0225	0.98785D 00	0.31562D 01
1.9990	0.46116D-00	0.39705D 02	2.0250	0.99383D 00	0.17420D 01
2.0000	0.50100D 00	0.39894D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000060

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.53746D 00	0.36257D 02
			2.0020	0.57351D 00	0.35798D 02
			2.0030	0.60896D 00	0.35051D 02
			2.0040	0.64352D 00	0.34035D 02
1.9725	0.60771D-02	0.15698D 01	2.0050	0.67694D 00	0.32774D 02
1.9750	0.11321D-01	0.27106D 01	2.0060	0.70900D 00	0.31298D 02
1.9775	0.20123D-01	0.44430D 01	2.0070	0.73948D 00	0.29641D 02
1.9800	0.34152D-01	0.69131D 01	2.0080	0.76823D 00	0.27838D 02
1.9825	0.55381D-01	0.10210D 02	2.0090	0.79512D 00	0.25928D 02
1.9850	0.85882D-01	0.14315D 02	2.0100	0.82006D 00	0.23948D 02
1.9860	0.10111D-00	0.16150D 02	2.0110	0.84301D 00	0.21936D 02
1.9870	0.11821D-00	0.18068D 02	2.0120	0.86394D 00	0.19927D 02
1.9880	0.13726D-00	0.20047D 02	2.0130	0.88287D 00	0.17951D 02
1.9890	0.15831D-00	0.22057D 02	2.0140	0.89986D 00	0.16037D 02
1.9900	0.18138D-00	0.24068D 02	2.0150	0.91497D 00	0.14208D 02
1.9910	0.20644D-00	0.26045D 02	2.0160	0.92831D 00	0.12484D 02
1.9920	0.23344D-00	0.27949D 02	2.0175	0.94523D 00	0.10122D 02
1.9930	0.26230D-00	0.29745D 02	2.0200	0.96626D 00	0.68443D 01
1.9940	0.29288D-00	0.31392D 02	2.0225	0.98014D 00	0.43933D 01
1.9950	0.32503D-00	0.32856D 02	2.0250	0.98884D 00	0.26770D 01
1.9960	0.35852D-00	0.34103D 02	2.0275	0.99402D 00	0.15484D 01
1.9970	0.39315D-00	0.35104D 02			
1.9980	0.42864D-00	0.35834D 02			
1.9990	0.46472D-00	0.36276D 02			
2.0000	0.50109D 00	0.36418D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000070

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.53485D 00	0.33588D 02
			2.0020	0.56828D 00	0.33221D 02
			2.0030	0.60122D 00	0.32625D 02
1.9700	0.56629D-02	0.13651D 01	2.0040	0.63345D 00	0.31812D 02
1.9725	0.10138D-01	0.22796D 01	2.0050	0.66478D 00	0.30798D 02
1.9750	0.17433D-01	0.36405D 01	2.0060	0.69499D 00	0.29604D 02
1.9775	0.28807D-01	0.55600D 01	2.0070	0.72393D 00	0.28254D 02
1.9800	0.45769D-01	0.81209D 01	2.0080	0.75146D 00	0.26774D 02
1.9825	0.69965D-01	0.11343D 02	2.0090	0.77744D 00	0.25190D 02
1.9850	0.10298D-00	0.15153D 02	2.0100	0.80181D 00	0.23532D 02
1.9860	0.11895D-00	0.16802D 02	2.0110	0.82449D 00	0.21826D 02
1.9870	0.13660D-00	0.18498D 02	2.0120	0.84546D 00	0.20100D 02
1.9880	0.15596D-00	0.20221D 02	2.0130	0.86469D 00	0.18378D 02
1.9890	0.17704D-00	0.21946D 02	2.0140	0.88222D 00	0.16685D 02
1.9900	0.19984D-00	0.23650D 02	2.0150	0.89808D 00	0.15039D 02
1.9910	0.22432D-00	0.25304D 02	2.0160	0.91232D 00	0.13460D 02
1.9920	0.25042D-00	0.26881D 02	2.0175	0.93083D 00	0.11244D 02
1.9930	0.27805D-00	0.28353D 02	2.0200	0.95480D 00	0.80401D 01
1.9940	0.30709D-00	0.29693D 02	2.0225	0.97158D 00	0.54978D 01
1.9950	0.33738D-00	0.30875D 02	2.0250	0.98282D 00	0.35953D 01
1.9960	0.36877D-00	0.31876D 02	2.0275	0.99002D 00	0.22485D 01
1.9970	0.40107D-00	0.32674D 02	2.0300	0.99443D 00	0.13448D 01
1.9980	0.43405D-00	0.33255D 02			
1.9990	0.46750D-00	0.33605D 02			
2.0000	0.50118D 00	0.33716D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000080

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.53276D 00	0.31432D 02
1.9675	0.51415D-02	0.11719D 01	2.0020	0.56406D 00	0.31131D 02
1.9700	0.89296D-02	0.19084D 01	2.0030	0.59496D 00	0.30641D 02
1.9725	0.14970D-01	0.29887D 01	2.0040	0.62528D 00	0.29971D 02
1.9750	0.24234D-01	0.45013D 01	2.0050	0.65485D 00	0.29132D 02
1.9775	0.37899D-01	0.65198D 01	2.0060	0.68350D 00	0.28141D 02
1.9800	0.57287D-01	0.90815D 01	2.0070	0.71108D 00	0.27014D 02
1.9825	0.83744D-01	0.12165D 02	2.0080	0.73749D 00	0.25770D 02
1.9840	0.10352D-00	0.14228D 02	2.0090	0.76259D 00	0.24431D 02
1.9850	0.11847D-00	0.15672D 02	2.0100	0.78632D 00	0.23017D 02
1.9860	0.13488D-00	0.17154D 02	2.0110	0.80861D 00	0.21549D 02
1.9870	0.15278D-00	0.18659D 02	2.0120	0.82941D 00	0.20050D 02
1.9880	0.17220D-00	0.20171D 02	2.0130	0.84870D 00	0.18539D 02
1.9890	0.19312D-00	0.21668D 02	2.0140	0.86649D 00	0.17034D 02
1.9900	0.21552D-00	0.23132D 02	2.0150	0.88278D 00	0.15555D 02
1.9910	0.23937D-00	0.24541D 02	2.0160	0.89761D 00	0.14115D 02
1.9920	0.26458D-00	0.25873D 02	2.0170	0.91103D 00	0.12729D 02
1.9930	0.29108D-00	0.27108D 02	2.0185	0.92864D 00	0.10773D 02
1.9940	0.31876D-00	0.28225D 02	2.0200	0.94344D 00	0.89912D 01
1.9950	0.34749D-00	0.29205D 02	2.0225	0.96262D 00	0.64469D 01
1.9960	0.37712D-00	0.30031D 02	2.0250	0.97612D 00	0.44455D 01
1.9970	0.40749D-00	0.30687D 02	2.0275	0.98527D 00	0.29479D 01
1.9980	0.43843D-00	0.31162D 02	2.0300	0.99122D 00	0.18800D 01
1.9990	0.46975D-00	0.31448D 02	2.0325	0.99495D 00	0.11530D 01
2.0000	0.50126D 00	0.31539D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000090

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.53104D 00	0.29645D 02
			2.0020	0.56057D 00	0.29392D 02
			2.0030	0.58977D 00	0.28979D 02
1.9650	0.45888D-02	0.99837D 00	2.0040	0.61848D 00	0.28414D 02
1.9675	0.77811D-02	0.15944D 01	2.0050	0.64655D 00	0.27705D 02
1.9700	0.12785D-01	0.24593D 01	2.0060	0.67385D 00	0.26865D 02
1.9725	0.20361D-01	0.36640D 01	2.0070	0.70024D 00	0.25906D 02
1.9750	0.31442D-01	0.52725D 01	2.0080	0.72562D 00	0.24842D 02
1.9775	0.47097D-01	0.73282D 01	2.0090	0.74990D 00	0.23691D 02
1.9800	0.68462D-01	0.98378D 01	2.0100	0.77298D 00	0.22467D 02
1.9810	0.78856D-01	0.10961D 02	2.0110	0.79481D 00	0.21189D 02
1.9820	0.90404D-01	0.12144D 02	2.0120	0.81535D 00	0.19872D 02
1.9830	0.10316D-00	0.13381D 02	2.0130	0.83455D 00	0.18535D 02
1.9840	0.11718D-00	0.14662D 02	2.0140	0.85241D 00	0.17191D 02
1.9850	0.13250D-00	0.15976D 02	2.0150	0.86894D 00	0.15857D 02
1.9860	0.14914D-00	0.17312D 02	2.0160	0.88413D 00	0.14545D 02
1.9870	0.16712D-00	0.18655D 02	2.0170	0.89804D 00	0.13267D 02
1.9880	0.18645D-00	0.19992D 02	2.0180	0.91068D 00	0.12035D 02
1.9890	0.20710D-00	0.21306D 02	2.0190	0.92213D 00	0.10857D 02
1.9900	0.22905D-00	0.22580D 02	2.0200	0.93242D 00	0.97400D 01
1.9910	0.25224D-00	0.23797D 02	2.0210	0.94163D 00	0.86893D 01
1.9920	0.27662D-00	0.24942D 02	2.0225	0.95356D 00	0.72462D 01
1.9930	0.30209D-00	0.25997D 02	2.0250	0.96903D 00	0.52070D 01
1.9940	0.32857D-00	0.26946D 02	2.0275	0.97997D 00	0.36139D 01
1.9950	0.35594D-00	0.27775D 02	2.0300	0.98743D 00	0.24227D 01
1.9960	0.38408D-00	0.28471D 02	2.0325	0.99236D 00	0.15687D 01
1.9970	0.41284D-00	0.29023D 02	2.0350	0.99550D 00	0.98105D 00
1.9980	0.44207D-00	0.29421D 02			
1.9990	0.47163D-00	0.29660D 02			
2.0000	0.50134D 00	0.29735D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000100

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0010	0.52959D 00	0.28131D 02
			2.0020	0.55763D 00	0.27914D 02
1.9625	0.40475D-02	0.84660D 00	2.0030	0.58538D 00	0.27561D 02
1.9650	0.67310D-02	0.13310D 01	2.0040	0.61270D 00	0.27076D 02
1.9675	0.10880D-01	0.20283D 01	2.0050	0.63949D 00	0.26467D 02
1.9700	0.17098D-01	0.29957D 01	2.0060	0.66560D 00	0.25742D 02
1.9725	0.26130D-01	0.42885D 01	2.0070	0.69094D 00	0.24913D 02
1.9750	0.38848D-01	0.59502D 01	2.0080	0.71539D 00	0.23990D 02
1.9775	0.56207D-01	0.80018D 01	2.0090	0.73889D 00	0.22986D 02
1.9800	0.79172D-01	0.10430D 02	2.0100	0.76134D 00	0.21914D 02
1.9810	0.90131D-01	0.11495D 02	2.0110	0.78270D 00	0.20788D 02
1.9820	0.10218D-00	0.12606D 02	2.0120	0.80291D 00	0.19622D 02
1.9830	0.11536D-00	0.13755D 02	2.0130	0.82193D 00	0.18429D 02
1.9840	0.12970D-00	0.14934D 02	2.0140	0.83976D 00	0.17221D 02
1.9850	0.14523D-00	0.16134D 02	2.0150	0.85638D 00	0.16013D 02
1.9860	0.16197D-00	0.17342D 02	2.0160	0.87179D 00	0.14815D 02
1.9870	0.17991D-00	0.18549D 02	2.0170	0.88601D 00	0.13639D 02
1.9880	0.19906D-00	0.19740D 02	2.0180	0.89908D 00	0.12493D 02
1.9890	0.21938D-00	0.20903D 02	2.0190	0.91101D 00	0.11386D 02
1.9900	0.24085D-00	0.22024D 02	2.0200	0.92187D 00	0.10326D 02
1.9910	0.26341D-00	0.23090D 02	2.0210	0.93168D 00	0.93177D 01
1.9920	0.28701D-00	0.24086D 02	2.0225	0.94459D 00	0.79123D 01
1.9930	0.31156D-00	0.25000D 02	2.0250	0.96174D 00	0.58763D 01
1.9940	0.33698D-00	0.25820D 02	2.0275	0.97430D 00	0.42299D 01
1.9950	0.36316D-00	0.26533D 02	2.0300	0.98320D 00	0.29512D 01
1.9960	0.39001D-00	0.27130D 02	2.0325	0.98932D 00	0.19956D 01
1.9970	0.41738D-00	0.27602D 02	2.0350	0.99340D 00	0.13080D 01
1.9980	0.44517D-00	0.27942D 02	2.0375	0.99604D 00	0.83088D 00
1.9990	0.47322D-00	0.28146D 02			
2.0000	0.50141D 00	0.28209D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000200

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.54181D 00	0.19837D 02
1.9500	0.62990D-02	0.88754D 00	2.0040	0.58121D 00	0.19532D 02
1.9550	0.12386D-01	0.16051D 01	2.0060	0.61981D 00	0.19040D 02
1.9600	0.23024D-01	0.27268D 01	2.0080	0.65726D 00	0.18376D 02
1.9650	0.40496D-01	0.43519D 01	2.0100	0.69322D 00	0.17559D 02
1.9700	0.67462D-01	0.65248D 01	2.0120	0.72741D 00	0.16611D 02
1.9730	0.89318D-01	0.80735D 01	2.0140	0.75959D 00	0.15558D 02
1.9760	0.11605D-00	0.97677D 01	2.0160	0.78958D 00	0.14426D 02
1.9780	0.13676D-00	0.10953D 02	2.0180	0.81726D 00	0.13244D 02
1.9800	0.15987D-00	0.12159D 02	2.0200	0.84255D 00	0.12038D 02
1.9820	0.18540D-00	0.13364D 02	2.0220	0.86541D 00	0.10833D 02
1.9840	0.21331D-00	0.14542D 02	2.0240	0.88589D 00	0.96512D 01
1.9860	0.24353D-00	0.15667D 02	2.0260	0.90405D 00	0.85130D 01
1.9880	0.27593D-00	0.16711D 02	2.0280	0.91998D 00	0.74342D 01
1.9900	0.31030D-00	0.17647D 02	2.0300	0.93383D 00	0.64276D 01
1.9920	0.34643D-00	0.18450D 02	2.0350	0.96037D 00	0.42765D 01
1.9940	0.38400D-00	0.19097D 02	2.0400	0.97752D 00	0.26729D 01
1.9960	0.42270D-00	0.19571D 02	2.0450	0.98793D 00	0.15694D 01
1.9980	0.46216D-00	0.19857D 02	2.0500	0.99388D 00	0.86563D 00
2.0000	0.50199D 00	0.19946D 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000300

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.534970 00	0.162240 02
			2.0040	0.567270 00	0.160540 02
1.9350	0.405580-02	0.489670-00	2.0060	0.599120 00	0.157810 02
1.9400	0.727740-02	0.823260 00	2.0080	0.630320 00	0.154090 02
1.9450	0.125730-01	0.132760 01	2.0100	0.660690 00	0.149460 02
1.9500	0.209230-01	0.205360 01	2.0120	0.690050 00	0.144010 02
1.9550	0.335560-01	0.304700 01	2.0140	0.718250 00	0.137840 02
1.9600	0.518890-01	0.433650 01	2.0160	0.745150 00	0.131050 02
1.9650	0.774130-01	0.591970 01	2.0180	0.770640 00	0.123770 02
1.9700	0.111500-00	0.775120 01	2.0200	0.794630 00	0.116110 02
1.9720	0.127780-00	0.853360 01	2.0220	0.817070 00	0.108210 02
1.9740	0.145650-00	0.933250 01	2.0240	0.837900 00	0.100170 02
1.9760	0.165120-00	0.101380 02	2.0260	0.857130 00	0.921200 01
1.9780	0.186200-00	0.109410 02	2.0280	0.874760 00	0.841500 01
1.9800	0.208870-00	0.117280 02	2.0300	0.890810 00	0.763580 01
1.9820	0.233090-00	0.124890 02	2.0320	0.905320 00	0.688280 01
1.9840	0.258800-00	0.132100 02	2.0350	0.924350 00	0.581700 01
1.9860	0.285900-00	0.138800 02	2.0400	0.949400 00	0.425060 01
1.9880	0.314280-00	0.144880 02	2.0450	0.967350 00	0.297920 01
1.9900	0.343800-00	0.150210 02	2.0500	0.979690 00	0.200290 01
1.9920	0.374310-00	0.154710 02	2.0550	0.987820 00	0.129160 01
1.9940	0.405620-00	0.158280 02	2.0600	0.992970 00	0.798930 00
1.9960	0.437560-00	0.160860 02	2.0650	0.996090 00	0.474010-00
1.9980	0.469900-00	0.162400 02			
2.0000	0.502440 00	0.162860 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000400

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.531000 00	0.140610 02
			2.0040	0.559020 00	0.139490 02
			2.0060	0.586750 00	0.137690 02
1.9250	0.409130-02	0.427380-00	2.0080	0.614050 00	0.135240 02
1.9300	0.679970-02	0.671490 00	2.0100	0.640800 00	0.132160 02
1.9350	0.109840-01	0.102260 01	2.0120	0.666880 00	0.128510 02
1.9400	0.172520-01	0.150930 01	2.0140	0.692170 00	0.124340 02
1.9450	0.263500-01	0.215920 01	2.0160	0.716590 00	0.119710 02
1.9500	0.391530-01	0.299390 01	2.0180	0.740030 00	0.114670 02
1.9550	0.566150-01	0.402360 01	2.0200	0.762430 00	0.109290 02
1.9600	0.797030-01	0.524110 01	2.0220	0.783730 00	0.103650 02
1.9630	0.966280-01	0.605050 01	2.0240	0.803880 00	0.978130 01
1.9660	0.116050-00	0.690680 01	2.0260	0.822850 00	0.918410 01
1.9680	0.130460-00	0.749700 01	2.0280	0.840610 00	0.858040 01
1.9700	0.146050-00	0.809700 01	2.0300	0.857170 00	0.797640 01
1.9720	0.162850-00	0.870140 01	2.0320	0.872520 00	0.737800 01
1.9740	0.180850-00	0.930430 01	2.0340	0.886690 00	0.679040 01
1.9760	0.200060-00	0.989940 01	2.0360	0.899690 00	0.621840 01
1.9780	0.220440-00	0.104800 02	2.0380	0.911570 00	0.566620 01
1.9800	0.241960-00	0.110390 02	2.0400	0.922370 00	0.513730 01
1.9820	0.264580-00	0.115700 02	2.0450	0.944980 00	0.393410 01
1.9840	0.288220-00	0.120670 02	2.0500	0.962030 00	0.292000 01
1.9860	0.312820-00	0.125220 02	2.0550	0.974500 00	0.210070 01
1.9880	0.338280-00	0.129290 02	2.0600	0.983340 00	0.146470 01
1.9900	0.364500-00	0.132820 02	2.0650	0.989420 00	0.989870 00
1.9920	0.391370-00	0.135780 02	2.0700	0.993460 00	0.648390 00
1.9940	0.418770-00	0.138110 02	2.0750	0.996080 00	0.411640-00
1.9960	0.446570-00	0.139770 02			
1.9980	0.474630-00	0.140760 02			
2.0000	0.502820 00	0.141040 02			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000500

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9200	0.58386D-02	0.52479D 00	2.0020	0.52836D 00	0.12583D 02
1.9250	0.90479D-02	0.77217D 00	2.0040	0.55345D 00	0.12501D 02
1.9300	0.13708D-01	0.11081D 01	2.0060	0.57833D 00	0.12371D 02
1.9350	0.20307D-01	0.15509D 01	2.0080	0.60290D 00	0.12193D 02
1.9400	0.29423D-01	0.21172D 01	2.0100	0.62707D 00	0.11969D 02
1.9450	0.41705D-01	0.28187D 01	2.0120	0.65075D 00	0.11703D 02
1.9500	0.57844D-01	0.36601D 01	2.0140	0.67386D 00	0.11397D 02
1.9550	0.78530D-01	0.46354D 01	2.0160	0.69632D 00	0.11055D 02
1.9580	0.93390D-01	0.52775D 01	2.0180	0.71806D 00	0.10680D 02
1.9600	0.10439D-00	0.57256D 01	2.0200	0.73902D 00	0.10277D 02
1.9620	0.11630D-00	0.61869D 01	2.0220	0.75915D 00	0.98491D 01
1.9640	0.12915D-00	0.66587D 01	2.0240	0.77840D 00	0.94016D 01
1.9660	0.14294D-00	0.71379D 01	2.0260	0.79674D 00	0.89387D 01
1.9680	0.15770D-00	0.76210D 01	2.0280	0.81415D 00	0.84646D 01
1.9700	0.17342D-00	0.81043D 01	2.0300	0.83060D 00	0.79837D 01
1.9720	0.19011D-00	0.85839D 01	2.0320	0.84608D 00	0.75001D 01
1.9740	0.20776D-00	0.90556D 01	2.0340	0.86060D 00	0.70176D 01
1.9760	0.22633D-00	0.95151D 01	2.0360	0.87416D 00	0.65399D 01
1.9780	0.24580D-00	0.99580D 01	2.0380	0.88676D 00	0.60705D 01
1.9800	0.26615D-00	0.10380D 02	2.0400	0.89844D 00	0.56122D 01
1.9820	0.28731D-00	0.10776D 02	2.0420	0.90922D 00	0.51678D 01
1.9840	0.30923D-00	0.11144D 02	2.0440	0.91913D 00	0.47397D 01
1.9860	0.33186D-00	0.11477D 02	2.0470	0.93242D 00	0.41319D 01
1.9880	0.35512D-00	0.11774D 02	2.0500	0.94396D 00	0.35698D 01
1.9900	0.37893D-00	0.12029D 02	2.0550	0.95969D 00	0.27423D 01
1.9920	0.40320D-00	0.12242D 02	2.0600	0.97162D 00	0.20546D 01
1.9940	0.42786D-00	0.12408D 02	2.0650	0.98046D 00	0.15013D 01
1.9960	0.45281D-00	0.12526D 02	2.0700	0.98684D 00	0.10700D 01
1.9980	0.47794D-00	0.12596D 02	2.0750	0.99133D 00	0.74375D 00
2.0000	0.50315D 00	0.12614D 02	2.0800	0.99442D 00	0.50421D 00

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000600

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.52647D 00	0.11490D 02
			2.0040	0.54939D 00	0.11427D 02
1.9100	0.48098D-02	0.40320D-00	2.0060	0.57215D 00	0.11327D 02
1.9150	0.72448D-02	0.57982D 00	2.0080	0.59467D 00	0.11190D 02
1.9200	0.10708D-01	0.81661D 00	2.0100	0.61689D 00	0.11018D 02
1.9250	0.15533D-01	0.11264D 01	2.0120	0.63872D 00	0.10812D 02
1.9300	0.22115D-01	0.15217D 01	2.0140	0.66011D 00	0.10575D 02
1.9350	0.30910D-01	0.20133D 01	2.0160	0.68100D 00	0.10309D 02
1.9400	0.42421D-01	0.26088D 01	2.0180	0.70133D 00	0.10016D 02
1.9450	0.57176D-01	0.33108D 01	2.0200	0.72105D 00	0.96990D 01
1.9500	0.75700D-01	0.41151D 01	2.0220	0.74011D 00	0.93608D 01
1.9530	0.88828D-01	0.46420D 01	2.0240	0.75848D 00	0.90043D 01
1.9560	0.10358D-00	0.51972D 01	2.0260	0.77612D 00	0.86326D 01
1.9580	0.11436D-00	0.55805D 01	2.0280	0.79300D 00	0.82486D 01
1.9600	0.12591D-00	0.59721D 01	2.0300	0.80911D 00	0.78556D 01
1.9620	0.13825D-00	0.63699D 01	2.0320	0.82442D 00	0.74563D 01
1.9640	0.15139D-00	0.67717D 01	2.0340	0.83893D 00	0.70538D 01
1.9660	0.16534D-00	0.71748D 01	2.0360	0.85264D 00	0.66509D 01
1.9680	0.18009D-00	0.75766D 01	2.0380	0.86554D 00	0.62501D 01
1.9700	0.19564D-00	0.79743D 01	2.0400	0.87764D 00	0.58539D 01
1.9720	0.21198D-00	0.83649D 01	2.0420	0.88896D 00	0.54645D 01
1.9740	0.22909D-00	0.87455D 01	2.0440	0.89951D 00	0.50841D 01
1.9760	0.24695D-00	0.91130D 01	2.0460	0.90930D 00	0.47145D 01
1.9780	0.26553D-00	0.94643D 01	2.0480	0.91837D 00	0.43571D 01
1.9800	0.28480D-00	0.97965D 01	2.0500	0.92674D 00	0.40135D 01
1.9820	0.30471D-00	0.10107D 02	2.0550	0.94478D 00	0.32210D 01
1.9840	0.32521D-00	0.10392D 02	2.0600	0.95912D 00	0.25317D 01
1.9860	0.34625D-00	0.10649D 02	2.0650	0.97028D 00	0.19489D 01
1.9880	0.36779D-00	0.10877D 02	2.0700	0.97878D 00	0.14693D 01
1.9900	0.38974D-00	0.11073D 02	2.0750	0.98513D 00	0.10849D 01
1.9920	0.41205D-00	0.11235D 02	2.0800	0.98977D 00	0.78458D 00
1.9940	0.43466D-00	0.11361D 02	2.0850	0.99310D 00	0.55568D 00
1.9960	0.45747D-00	0.11450D 02	2.0900	0.99543D 00	0.38545D-00
1.9980	0.48043D-00	0.11502D 02			
2.0000	0.50345D 00	0.11515D 02			

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TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000700

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.9050	0.57129D-02	0.43504D-00	2.0020	0.52504D 00	0.10640D 02
1.9100	0.82925D-02	0.60455D 00	2.0040	0.54627D 00	0.10589D 02
1.9150	0.11844D-01	0.82524D 00	2.0060	0.56738D 00	0.10509D 02
1.9200	0.16646D-01	0.11066D 01	2.0080	0.58829D 00	0.10399D 02
1.9250	0.23026D-01	0.14575D 01	2.0100	0.60895D 00	0.10261D 02
1.9300	0.31350D-01	0.18859D 01	2.0120	0.62932D 00	0.10096D 02
1.9350	0.42022D-01	0.23969D 01	2.0140	0.64932D 00	0.99054D 01
1.9400	0.55460D-01	0.29924D 01	2.0160	0.66892D 00	0.96906D 01
1.9450	0.72083D-01	0.36699D 01	2.0180	0.68807D 00	0.94534D 01
1.9500	0.92282D-01	0.44210D 01	2.0200	0.70672D 00	0.91957D 01
1.9520	0.10144D-00	0.47391D 01	2.0220	0.72484D 00	0.89196D 01
1.9540	0.11124D-00	0.50657D 01	2.0240	0.74239D 00	0.86270D 01
1.9560	0.12171D-00	0.53993D 01	2.0260	0.75934D 00	0.83202D 01
1.9580	0.13284D-00	0.57384D 01	2.0280	0.77566D 00	0.80014D 01
1.9600	0.14466D-00	0.60814D 01	2.0300	0.79134D 00	0.76729D 01
1.9620	0.15717D-00	0.64266D 01	2.0320	0.80635D 00	0.73369D 01
1.9640	0.17037D-00	0.67720D 01	2.0340	0.82068D 00	0.69956D 01
1.9660	0.18426D-00	0.71156D 01	2.0360	0.83433D 00	0.66512D 01
1.9680	0.19883D-00	0.74553D 01	2.0380	0.84729D 00	0.63057D 01
1.9700	0.21408D-00	0.77889D 01	2.0400	0.85955D 00	0.59610D 01
1.9720	0.22998D-00	0.81142D 01	2.0420	0.87113D 00	0.56192D 01
1.9740	0.24653D-00	0.84290D 01	2.0440	0.88203D 00	0.52818D 01
1.9760	0.26369D-00	0.87311D 01	2.0460	0.89226D 00	0.49505D 01
1.9780	0.28144D-00	0.90182D 01	2.0480	0.90184D 00	0.46268D 01
1.9800	0.29975D-00	0.92881D 01	2.0500	0.91078D 00	0.43119D 01
1.9820	0.31858D-00	0.95389D 01	2.0520	0.91909D 00	0.40069D 01
1.9840	0.33789D-00	0.97685D 01	2.0550	0.93045D 00	0.35703D 01
1.9860	0.35764D-00	0.99750D 01	2.0600	0.94660D 00	0.29040D 01
1.9880	0.37777D-00	0.10157D 02	2.0650	0.95963D 00	0.23202D 01
1.9900	0.39825D-00	0.10313D 02	2.0700	0.96995D 00	0.18210D 01
1.9920	0.41901D-00	0.10441D 02	2.0750	0.97798D 00	0.14039D 01
1.9940	0.43999D-00	0.10540D 02	2.0800	0.98411D 00	0.10632D 01
1.9960	0.46115D-00	0.10611D 02	2.0850	0.98872D 00	0.79089D 00
1.9980	0.48242D-00	0.10651D 02	2.0900	0.99212D 00	0.57794D 00
2.0000	0.50373D 00	0.10661D 02	2.0950	0.99458D 00	0.41485D-00

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.000800

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.52392D 00	0.99546D 01
			2.0040	0.54379D 00	0.99124D 01
			2.0060	0.56355D 00	0.98458D 01
1.9000	0.63918D-02	0.44952D-00	2.0080	0.58316D 00	0.97551D 01
1.9050	0.90210D-02	0.60884D 00	2.0100	0.60256D 00	0.96412D 01
1.9100	0.12553D-01	0.81184D 00	2.0120	0.62171D 00	0.95048D 01
1.9150	0.17224D-01	0.10658D 01	2.0140	0.64056D 00	0.93470D 01
1.9200	0.23307D-01	0.13774D 01	2.0160	0.65908D 00	0.91688D 01
1.9250	0.31104D-01	0.17526D 01	2.0180	0.67723D 00	0.89716D 01
1.9300	0.40945D-01	0.21954D 01	2.0200	0.69496D 00	0.87567D 01
1.9350	0.53173D-01	0.27074D 01	2.0220	0.71224D 00	0.85256D 01
1.9400	0.68132D-01	0.32871D 01	2.0240	0.72905D 00	0.82798D 01
1.9450	0.86149D-01	0.39291D 01	2.0260	0.74535D 00	0.80211D 01
1.9480	0.98548D-01	0.43404D 01	2.0280	0.76113D 00	0.77511D 01
1.9500	0.10751D-00	0.46237D 01	2.0300	0.77635D 00	0.74715D 01
1.9520	0.11705D-00	0.49132D 01	2.0320	0.79101D 00	0.71840D 01
1.9540	0.12717D-00	0.52079D 01	2.0340	0.80508D 00	0.68903D 01
1.9560	0.13788D-00	0.55064D 01	2.0360	0.81857D 00	0.65921D 01
1.9580	0.14919D-00	0.58075D 01	2.0380	0.83145D 00	0.62911D 01
1.9600	0.16111D-00	0.61097D 01	2.0400	0.84373D 00	0.59888D 01
1.9620	0.17363D-00	0.64117D 01	2.0420	0.85540D 00	0.56868D 01
1.9640	0.18676D-00	0.67118D 01	2.0440	0.86648D 00	0.53866D 01
1.9660	0.20048D-00	0.70084D 01	2.0460	0.87695D 00	0.50895D 01
1.9680	0.21479D-00	0.72998D 01	2.0480	0.88684D 00	0.47967D 01
1.9700	0.22967D-00	0.75844D 01	2.0500	0.89614D 00	0.45095D 01
1.9720	0.24512D-00	0.78604D 01	2.0520	0.90488D 00	0.42290D 01
1.9740	0.26111D-00	0.81261D 01	2.0540	0.91306D 00	0.39559D 01
1.9760	0.27762D-00	0.83798D 01	2.0570	0.92434D 00	0.35624D 01
1.9780	0.29462D-00	0.86198D 01	2.0600	0.93446D 00	0.31900D 01
1.9800	0.31209D-00	0.88446D 01	2.0650	0.94896D 00	0.26208D 01
1.9820	0.32999D-00	0.90526D 01	2.0700	0.96078D 00	0.21198D 01
1.9840	0.34828D-00	0.92424D 01	2.0750	0.97027D 00	0.16881D 01
1.9860	0.36694D-00	0.94126D 01	2.0800	0.97777D 00	0.13234D 01
1.9880	0.38592D-00	0.95620D 01	2.0850	0.98361D 00	0.10214D 01
1.9900	0.40518D-00	0.96895D 01	2.0900	0.98808D 00	0.77611D 00
1.9920	0.42466D-00	0.97942D 01	2.0950	0.99146D 00	0.58058D 00
1.9940	0.44434D-00	0.98753D 01	2.1000	0.99396D 00	0.42759D-00
1.9960	0.46415D-00	0.99323D 01			
1.9980	0.48405D-00	0.99646D 01			
2.0000	0.50399D 00	0.99721D 01			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION .

ETA = 0.000900

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
			2.0020	0.52302D 00	0.93864D 01
1.8900	0.49141D-02	0.33556D-00	2.0040	0.54176D 00	0.93505D 01
1.8950	0.68702D-02	0.45173D-00	2.0060	0.56041D 00	0.92941D 01
1.9000	0.94845D-02	0.59974D 00	2.0080	0.57893D 00	0.92175D 01
1.9050	0.12930D-01	0.78526D 00	2.0100	0.59727D 00	0.91212D 01
1.9100	0.17409D-01	0.10140D 01	2.0120	0.61540D 00	0.90060D 01
1.9150	0.23151D-01	0.12913D 01	2.0140	0.63328D 00	0.88724D 01
1.9200	0.30411D-01	0.16217D 01	2.0160	0.65088D 00	0.87214D 01
1.9250	0.39463D-01	0.20087D 01	2.0180	0.66815D 00	0.85540D 01
1.9300	0.50594D-01	0.24536D 01	2.0200	0.68508D 00	0.83712D 01
1.9350	0.64095D-01	0.29558D 01	2.0220	0.70163D 00	0.81740D 01
1.9400	0.80242D-01	0.35117D 01	2.0240	0.71777D 00	0.79639D 01
1.9450	0.99290D-01	0.41146D 01	2.0260	0.73348D 00	0.77419D 01
1.9480	0.11220D-00	0.44949D 01	2.0280	0.74873D 00	0.75094D 01
1.9500	0.12145D-00	0.47545D 01	2.0300	0.76351D 00	0.72677D 01
1.9520	0.13122D-00	0.50179D 01	2.0320	0.77780D 00	0.70181D 01
1.9540	0.14152D-00	0.52842D 01	2.0340	0.79158D 00	0.67621D 01
1.9560	0.15236D-00	0.55523D 01	2.0360	0.80484D 00	0.65010D 01
1.9580	0.16373D-00	0.58211D 01	2.0380	0.81758D 00	0.62361D 01
1.9600	0.17564D-00	0.60893D 01	2.0400	0.82978D 00	0.59687D 01
1.9620	0.18809D-00	0.63557D 01	2.0420	0.84145D 00	0.57001D 01
1.9640	0.20106D-00	0.66191D 01	2.0440	0.85258D 00	0.54315D 01
1.9660	0.21456D-00	0.68781D 01	2.0460	0.86318D 00	0.51641D 01
1.9680	0.22857D-00	0.71313D 01	2.0480	0.87324D 00	0.48990D 01
1.9700	0.24308D-00	0.73775D 01	2.0500	0.88278D 00	0.46371D 01
1.9720	0.25808D-00	0.76152D 01	2.0520	0.89179D 00	0.43795D 01
1.9740	0.27354D-00	0.78431D 01	2.0540	0.90030D 00	0.41270D 01
1.9760	0.28944D-00	0.80600D 01	2.0570	0.91213D 00	0.37597D 01
1.9780	0.30577D-00	0.82644D 01	2.0600	0.92287D 00	0.34079D 01
1.9800	0.32249D-00	0.84553D 01	2.0650	0.93852D 00	0.28613D 01
1.9820	0.33958D-00	0.86313D 01	2.0700	0.95158D 00	0.23692D 01
1.9840	0.35701D-00	0.87915D 01	2.0750	0.96231D 00	0.19347D 01
1.9860	0.37473D-00	0.89347D 01	2.0800	0.97102D 00	0.15581D 01
1.9880	0.39273D-00	0.90601D 01	2.0850	0.97799D 00	0.12375D 01
1.9900	0.41096D-00	0.91670D 01	2.0900	0.98348D 00	0.96936D 00
1.9920	0.42939D-00	0.92544D 01	2.0950	0.98776D 00	0.74882D 00
1.9940	0.44797D-00	0.93220D 01	2.1000	0.99104D 00	0.57047D 00
1.9960	0.46666D-00	0.93693D 01	2.1050	0.99353D 00	0.42861D-00
1.9980	0.48543D-00	0.93958D 01	2.1100	0.99538D 00	0.31759D-00
2.0000	0.50423D 00	0.94016D 01			

TABLE FOR THE EVALUATION OF THE FAXEN APPROXIMATION TO THE SOLUTION OF THE LAMM EQUATION

ETA = 0.001000

Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY	Y	PSI(Y,ETA)	DPSI(Y,ETA)/DY
1.8850	0.52342D-02	0.33673D-00	2.0020	0.522290 00	0.89056D 01
1.8900	0.71791D-02	0.44550D-00	2.0040	0.54007D 00	0.88745D 01
1.8950	0.97356D-02	0.58210D 00	2.0060	0.55777D 00	0.88258D 01
1.9000	0.13054D-01	0.75112D 00	2.0080	0.57536D 00	0.87599D 01
1.9050	0.17309D-01	0.95719D 00	2.0100	0.59280D 00	0.86771D 01
1.9100	0.22695D-01	0.12047D 01	2.0120	0.61006D 00	0.85779D 01
1.9150	0.29430D-01	0.14973D 01	2.0140	0.62710D 00	0.84629D 01
1.9200	0.37748D-01	0.18378D 01	2.0160	0.64390D 00	0.83328D 01
1.9250	0.47891D-01	0.22278D 01	2.0180	0.66043D 00	0.81882D 01
1.9300	0.60108D-01	0.26671D 01	2.0200	0.67665D 00	0.80302D 01
1.9350	0.74640D-01	0.31533D 01	2.0220	0.69254D 00	0.78594D 01
1.9400	0.91712D-01	0.36818D 01	2.0240	0.70808D 00	0.76769D 01
1.9450	0.11152D-00	0.42455D 01	2.0260	0.72324D 00	0.74837D 01
1.9480	0.12478D-00	0.45967D 01	2.0280	0.73800D 00	0.72807D 01
1.9500	0.13421D-00	0.48348D 01	2.0300	0.75235D 00	0.70691D 01
1.9520	0.14412D-00	0.50750D 01	2.0320	0.76628D 00	0.68500D 01
1.9540	0.15451D-00	0.53165D 01	2.0340	0.77975D 00	0.66244D 01
1.9560	0.16539D-00	0.55583D 01	2.0360	0.79277D 00	0.63934D 01
1.9580	0.17674D-00	0.57996D 01	2.0380	0.80532D 00	0.61581D 01
1.9600	0.18858D-00	0.60392D 01	2.0400	0.81740D 00	0.59197D 01
1.9620	0.20090D-00	0.62762D 01	2.0420	0.82900D 00	0.56791D 01
1.9640	0.21368D-00	0.65095D 01	2.0440	0.84011D 00	0.54374D 01
1.9660	0.22693D-00	0.67379D 01	2.0460	0.85075D 00	0.51956D 01
1.9680	0.24063D-00	0.69604D 01	2.0480	0.86090D 00	0.49546D 01
1.9700	0.25477D-00	0.71759D 01	2.0500	0.87057D 00	0.47154D 01
1.9720	0.26933D-00	0.73833D 01	2.0520	0.87976D 00	0.44788D 01
1.9740	0.28430D-00	0.75816D 01	2.0540	0.88848D 00	0.42455D 01
1.9760	0.29965D-00	0.77695D 01	2.0570	0.90071D 00	0.39035D 01
1.9780	0.31537D-00	0.79463D 01	2.0600	0.91192D 00	0.35730D 01
1.9800	0.33143D-00	0.81108D 01	2.0650	0.92846D 00	0.30524D 01
1.9820	0.34780D-00	0.82622D 01	2.0700	0.94251D 00	0.25753D 01
1.9840	0.36447D-00	0.83997D 01	2.0750	0.95430D 00	0.21458D 01
1.9860	0.38139D-00	0.85223D 01	2.0800	0.96406D 00	0.17657D 01
1.9880	0.39855D-00	0.86295D 01	2.0850	0.97204D 00	0.14349D 01
1.9900	0.41590D-00	0.87206D 01	2.0900	0.97848D 00	0.11516D 01
1.9920	0.43342D-00	0.87950D 01	2.0950	0.98363D 00	0.91277D 00
1.9940	0.45107D-00	0.88523D 01	2.1000	0.98768D 00	0.71447D 00
1.9960	0.46881D-00	0.88922D 01	2.1050	0.99083D 00	0.55231D 00
1.9980	0.48662D-00	0.89145D 01	2.1100	0.99325D 00	0.42165D-00
2.0000	0.50446D 00	0.89189D 01	2.1150	0.99509D 00	0.31790D-00

Thanks are due to Dr. Robert Jastrow, Director, Goddard Institute for Space Studies, for use of the Institute's computing facilities.

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(Paper 70B1-171)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

Publications of the National Bureau of Standards*

Selected Abstracts

A new differential operator of the pure wave type, J. E. Lagnese, *J. Different. Eq.* **1,** *No. 2, 171–187 (Apr. 1965).*
Consider the differential operator $\mathcal{L}u = u + c(t)u$ $(= \partial_t^2 - \sum_{i=1}^{7} \partial_{x_i}^2)$ with $c(t)$ analytic on some open, connected interval of the real t-axis. Under this requirement, we seek to determine explicitly the form of $c(t)$ for which $\mathcal{L}$ satisfies Huygen's principle in the sense of "Hadamard's premise." (Mathisson called such operators "pure wave".) *Theorem. $\mathcal{L}$ is of pure waves if, and only if, it is equivalent to one of the following operators:* u, $u - \frac{2}{t^2}u$, $u - \frac{6}{t^2}u$, $u - \frac{6t(2+t^3)}{(1-t^3)^2}u$.
This theorem extends earlier results along this line and presents a new, and substantially more complicated, counterexample to "Hadamard's conjecture," namely the operator $u - 6t(2+t^3)/(1-t^3)^2\, u$.

Aggregation in matrix models of resource flows, D. Rosenblatt, *Am. Stat.* **19,** *No. 3, 36–39 (June 1965).*
The aggregation of finite substochastic systems of the form $x(I-A)=w$, A a square and substochastic matrix and w a nonnegative vector, is considered from normative and descriptive standpoints in connection with models of resource flows. Certain of the fundamental functional equations of the theory of aggregation are derived and are given a graph theoretic interpretation.

An application of the inverse Z-transform theory to the synthesis of linear antenna arrays, M. T. Ma, *IEEE Trans. Ant. Prop.* **AP-12,** *No. 6, 798–799 (Nov. 1964).*
It has been known that the Z-transform theory can be applied as a new technique for analyzing linear array problems. The reverse problem, synthesizing linear arrays by applying the inverse Z-transform theory, has not yet been reported in the literature.
This communication demonstrates that the job of synthesizing an antenna array, when certain conditions are satisfied by the given radiation pattern, can be accomplished particularly as a result of applying the inverse Z-transform theory originally developed for sampled-data systems, and that the error occurring between the synthesized and specified patterns can be estimated.

Spacetime coordinate systems, G. E. Hudson, *Proc. Intern. Conf. Chronometry, pp. 197-221 (Lausanne, Switzerland, June 1964).*
The principle of covariance is sometimes said to preclude the necessity for specifying a set of coordinates operationally. A simple example illustrates that this is not the case. This leads to the basic problem of specifying coordinate systems of physical use and interest over the vast reaches of spacetime. Examples of two operational definitions for spacetime networks associated with the surface of a spinning sphere serve to make definite some of the requirements, and show the importance and use of the general Doppler effect. For space navigational purposes, the gravitational fields, medium refraction and dispersion, and instrument uncertainty are discussed.
The point of view we have adopted here stems from, but is not limited to, a consideration of general relativistic effects and uses much of the formalism of relativity theory. It suggests moreover, a new general approach to time synchronization, space communication, and tracking problems, which is reminiscent of the concept of "homologue" space introduced by various authors to eliminate major refraction effects in a medium, or of Fock's concept of harmonic coordinates. Of more importance is the demonstration that a generalized range and range-rate coordinate system introduced here can be identified with a class of "null" coordinate systems previously mentioned by Synge.

Correlated walk and diffusion equations in a driving force, J. R. Manning, *Phys. Rev.* **139,** *No. 1A, A126–A135 (July 5, 1965).*
The correlation factor for diffusion by a vacancy mechanism arises because an atom can exchange with a given vacancy more than once. This results in a series of correlated exchanges between the atom and vacancy. In order to include correlation effects in the diffusion equations, one must allow the atom sufficient time to complete any corrrlated series of jumps. With this in mind, the basic diffusion equations are derived in terms of the number N of correlated series. A complete description of planar diffusion then can be presented in terms of effective jump frequencies, where each jump is independent of previous jumps. The resulting equations are correct to first order in small quantities. This is shown to be true even where there are driving forces and a diffusion coefficient gradient. Final expressions for the diffusion coefficient, correlation factor, drift velocity, and diffusion flux are valid for diffusion in any direction in cubic crystals and along particular directions in noncubic crystals. As an example, effects from a chemical concentration gradient are discussed.

Generalized master equation for quantum mechanical systems to all orders in the density, J. Weinstock, *Phys. Rev.* **136,** *2d series, No. 4A, A879–A888 (Nov. 1964).*
An exact generalized master equation is derived for a large quantum mechanical system in the form of a power series in the density. This derivation is a quantum mechanical generalization of a previous work by the author for classical systems. The quantum equation can be viewed as a time-dependent analog of the virial expansion of the quantum mechanical partition function for both degenerate systems (B.E. or F.D. statistics) and non-degenerate systems. The coefficients of the series, in the quantum equation, are explicitly given in terms of operators (Green functions) which are determined by the dynamics of isolated groups of particles and are convergent functions of the interaction potential. Equations are obtained for the off-diagonal elements of the density matrix as well as for the diagonal elements. The equation for the diagonal elements is shown to reduce to a Markoffian master equation for asymptotically long times, and an explicit expression is obtained for the "scattering" operator of this asymptotic equation.

Master equations and Markov processes, I. Oppenheim and K. E. Shuler, *Phys. Rev.* **138,** *No. 4B, B1117–B1011 (May 24, 1965).*
The processes described by generalized master equations (GME), derived from the Liouville equation on the basis of various physical and dynamic arguments, have been termed Markovian or non-Markovian depending upon whether the GME did not or did involve an explicit time integration. We have shown here that these designations are not in accord with the (very specific) mathematical definition of a Markov and a non-Markovian process. We suggest that the much more appropriate terms Pauli process (or equation) and non-Paulian process (or equation) be used instead to designate processes described by the GME.

Analogies between theories of antenna arrays and passive networks, M. T. Ma., *IEEE Intern. Conv. Record* **13,** *Pt. 5, 150–154 (Mar. 1965).*
This paper demonstrates the analogy between two different fields, antenna arrays and passive networks, which are generally not known to each other. Successful application, either directly or with modification, of various known techniques for synthesizing a passive network, when an impulse response is specified, to the synthesis of linear antenna arrays with amplitude excitations, phase distributions and element spacings as the controlling parameters, when a desired radiation pattern is given, is presented with many interesting examples.

Approximations to the distribution of quadratic forms, M. M. Siddiqui, *Ann. Math. Stat.* **36,** *No. 2, 677–682 (Apr. 1965).*
Let $Q = 1/2 \sum_{j=1}^{2k} a_j X_j^2$, where $0 < a_1 \leq a_2 \leq \ldots \leq a_{2k}$ are constants and $X_1, \ldots, X_{2k}$ are independent $N(0, 1)$ variates. If a_j's are equal within groups of even sizes the distribution of Q can be evaluated exactly: otherwise some approximations to this distribution are needed. In the present paper an approximation to $F(x) = Pr(Q > x)$ is obtained by bounding Q by Q_1 and Q_2, where Q_1 and Q_2 are quadratic forms whose exact distributions can be evaluated. Let $F_i(x) = Pr(Q_i > x)$, $i = 1, 2$, and let $\hat{F}(x) = (1-\theta) F_1(x) + \theta F_2(x)$. The approximation to $F(x)$ is obtained by minimizing $d(F, \hat{F})$ for variations of θ where $d(\ldots)$ is the distance function of the metric space $L^2(0, \infty)$.

Estimation for a one-parameter exponential model, J. A. Speckman and R. G. Cornell, *J. Amer. Stat. Assoc.* **60,** *560–572 (1965).*
The partial totals estimation procedure is presented, illustrated and evaluated for the model $y = \exp(-\rho t) + e$ when the values of t are equally spaced. Tables of estimates using this estimation procedure are given for the case where the smallest value of t is zero. The evaluation consists of an analytical investigation of the large sample properties of the partial totals estimator and a comparative study of its small sample properties relative to three other estimators based on Monte Carlo results where it is assumed either that y is a binomial proportion or that y is a proportion with variance proportional to a binomial variance.

Inequalities for permanents and permanental minors, R. A. Brualdi and M. Newman, *Proc. Cambridge Phil. Soc.* **61,** *pt. 3, 741–746 (Jan. 26, 1965).*
The principal result of this paper is that for all non-negative doubly-stochastic matrices A and all α such that $0 \leq \alpha \leq 1$, $\mathrm{per}(\alpha I + (1-\alpha)A) \leq \alpha + (1-\alpha)\mathrm{per}(A)$, where per (A) denotes the permanent of A.

On the binary collision expansion of the classical *N*-body Green's function, J. Weinstock, *Phys. Rev.* **126,** *No. 1, 341–344 (Apr. 1, 1962).*
The explicit time integrations in the formal Binary Collision Expansion of the classical N-body Green's function are performed to obtain a product of binary collision operators which bears a strong resemblance to the Mayer product of f_{ij}'s. These integrations are exact for hard sphere interactions and are expected to be a good approximation for finite but short range pair interactions. The resulting expansion can be averaged over configuration space to obtain a transport equation for a dense gas in analogy with the cluster expansion of the classical partition function.

On matching problems, J. Edmonds, A. J. Goldman, C. Witzgall, C. T. Zahn, Jr., *Proc. ARO Working Group on Computer, ARO-D Report 65-1, pp. 46–50 (Office Chief Res. and Develop., Feb. 1964).*
Combinatorial optimization problems, although finite, remain intracable in practice so long as all known solution methods involve an amount of labor which increases exponentially with problem size. The present paper reports the recent development, by members of the Combinatorial Mathematics Project at the National Bureau of Standards, of efficient solution and checking algorithms for one class of problems. The prototype of this class is the problem of finding in a linear graph a set of pairwise disjoint edges (a "matching") with as many edges as possible; several generalizations are also discussed.

Paths, trees, and flowers, J. Edmonds, *Can. J. Math.* **17,** *449–467 (1965).*
A matching in a graph is a subset of its edges such that no two meet the same vertex. The theory of matchings is treated. In particular, an efficient algorithm is described for finding in a given graph a matching of maximum cardinality. The Konig theorem is generalized to arbitrary (non-bipartite) graphs.

Realization of semi-multipliers as multipliers, H. Fell and A. J. Goldman, *Am. Math. Mo.* **72,** *No. 6, 641 (June–July 1965).*
The semi-multipliers of a linear associative algebra A are linear transformations obeying a generalization of the formula imposed on the multipliers of A by the associative law. Buck showed that A could be imbedded as an ideal in a very large algebra B so that each semi-multiplier of A was realized as a multiplier in B, and asked how to construct a smaller algebra B with the same property. This note presents such a construction.

Some remarks on certain generalized Dedekind sums, H. Rademacher, *Acta Arithmetica* **9,** *Sec. 1, 97–105 (1964).*
Some years ago the German mathematician C. Meyer introduced a certain generalization of the Dedekind sums. This paper has as its aim to formulate the "true" generalization, which lies behind all the previous attempts. In this new formulation the theorems become not only general but also really transparent. A reciprocity theorem is to be expected in this theory and is indeed fully established. The paper closes with a hint at a useful Euclidean algorithm.
The paper is dedicated to L. J. Mordell (who himself has made contributions to the theory of Dedekind sums). It is an invited paper for an anniversary volume.

Stochastic theory of diffusion in a plasma across a magnetic field, C. M. Tchen, *Proc. Intern. Symp. Diffusion, Feldafing, Germany, June 29–July 3, 1964, pp. 118–123 (1964).*
A stochastic theory is proposed to investigate the diffusion across a constant magnetic field for the cases of collision and collective oscillations. A general formula is obtained, which shows the two cases separately, and the underlying assumptions permitting such a separation are clarified. It is found that the diffusion by collective oscillations is determined by the wave energy, while the collisional diffusion is determined by the thermal energy.

Electromagnetic properties of a quantized relativistic electron-positron gas, L. A. Steinert, *Il Nuovo Cimento, Serie X* **36,** *935–953 (Apr. 1965).*
A relativistic quantum statistical description is given of electromagnetic wave propagation in a uniformly magnetized electron-positron gas, with specific attention devoted to long wavelength propagation as determined by the "collective" approximation. Collisionless absorption effects derived from relativity theory are compared with those obtained from the corresponding nonrelativistic treatment, showing that the two theories predict substantially different effects even for "nonrelativistic" temperatures. Contrasts appear in the predicted characteristics of the cyclotron resonance absorption effect, including the fact that Landau damping is completely nonexistent in the relativistic gas for waves propagating at phase velocities exceeding the velocity of light.

Wave functions for anharmonic oscillators by perturbation methods, A. M. Shorb, R. Schroeder, and E. R. Lippincott, *J. Chem. Phys.* **37,** *1043 (Sept. 1, 1962).*
Approximate wave functions and the corresponding energies are given in tabular form for anharmonic potentials which can be expanded in polynomial series up to the sixth power in the displacement, x.

Localized-induction concept on a curved vortex and motion of an elliptic vortex ring, R. J. Arms and F. R. Hama, *Phys. Fluids* **8,** *No. 4, 553–559 (Apr. 1965).*
The localized-induction concept for the induction effect of a smooth curved vortex on itself is derived. This concept is applicable to the limiting case of a vortex filament of infinitesimal core size and of negligible long-distance effect, and was already successfully utilized in the investigations of the motion and deformation of a curved vortex filament given various initial configurations. Two theorems obtained under this concept are that the arc length of a vortex filament and the projected area of a closed vortex filament are both invariant with respect to time. These theoretical predictions are examined by a numerical analysis of the motion of an initially plane elliptic vortex ring of various eccentricities.

Random-walk model of chain-polymer adsorption at a surface, R. J. Rubin, *J. Chem. Phys.* **43,** *No. 7, 2392–2407 (Oct. 1, 1965).*
A random walk lattice model of adsorption of an isolated polymer chain at a solution surface is investigated. On neglecting the self-excluded volume, a number of one-dimensional characteristics of the monomer unit distribution are determined analytically in the limit of long polymer chains. On assuming that one end of the poly-

mer chain lies in the surface layer, the mean number of monomer units adsorbed in the surface layer $\nu(\theta, N)$ and the mean distance of the other end of the chain from the surface layer $z(\theta, N)$ are determined, where N is the number of monomer units in the chain and θ is the adsorption energy of each monomer unit in the surface layer measured in units of kT. The lattice models considered include the simple cubic, hexagonal, close-packed, face-centered cubic, and body-centered cubic lattices. In the limit in which $N \to \infty$, both $\nu(\theta, N)$ and $z(\theta, N)$ exhibit a very interesting discontinuity at a lattice-dependent adsorption energy θ_c. For example for $\theta > \theta_c$, $\nu(\theta, N)$ (which is also proportional to the average adsorption energy of a polymer chain) is proportional to N. For $\theta < \theta_c$, $\nu(\theta, N)$ is proportional to a constant of order unity; and for $\theta = \theta_c$, $\nu(\theta, N)$ is proportional to $N^{1/2}$. It is shown that the probability distribution of the end of the chain decreases exponentially with increasing distance from the surface layer for $\theta > \theta_c$.

In addition, the mean number of monomer units in the kth layer from the surface is determined for $N >> 1$ and $\theta > \theta_c$ and is found to decrease exponentially with increasing k. In effect, for $\theta > \theta_c$ the polymer chain exists in an adsorbed state. An improvement in the model, which includes short range correlation between successive steps in the random walk description, is also considered.

Interaction between configurations with several open shells, U. Fano, *Phys. Rev.* **140,** *No. 1A, A67–A75 (Oct. 4, 1965).*

Antisymmetrized product wave functions are written for atomic configurations with an arbitrary number of partially or entirely filled shells. Interaction matrix elements between such configurations are calculated and a numerical evaluation is given for the autoionization of an inner excitation level of neon. A suitable notation keeps the analytical complexities within reasonable bounds.

Other NBS Publications

J. Res. NBS 69A (Phys. and Chem.), No. 6 (Nov.-Dec. 1965), 75 cents.

Arc measurement of some argon transition probabilities. C. H. Popenoe and J. B. Shumaker, Jr.

Theoretical interpretation of the third spectrum of gold (Au III). Y. Shadmi.

Photopolarographic behavior of inorganic depolarizers. R. A. Durst and J. K. Taylor.

Spectral structure of critical opalescence: binary mixture. R. D. Mountain.

Irregularities in the NBS (1955) provisional temperature scale. H. M. Roder.

Isotherms determined by the National Bureau of Standards acoustical thermometer in the liquid helium temperature range. G. Cataland and H. Plumb.

Synthesis of D-glucose-3-^{14}C and related compounds. H. L. Frush, L. T. Sniegoski, N. B. Holt, and H. S. Isbell.

Correlation of large longitudinal deformations with different histories. L. J. Zapas and T. Craft.

Crystallography of tetracalcium phosphate. W. E. Brown and E. F. Epstein.

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I Session: Jupiter, as observed at long radio waves.
II Session: Jupiter, as observed at short radio waves.
III Session: Passive radio observations of Venus, Saturn, Mercury, Mars, and Uranus.
IV Session: Passive radio observations of the Moon.
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VI Session: Radar observations of the Moon.

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JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 70B, No. 1 January–March 1966
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Library of Congress Catalog Card Number: 62–4414

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